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Long range dependence, no arbitrage and the Black-Scholes formula

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Abstract A bond and stock model is considered where the driving process is the sum of a Wiener process W and a continuous process Z with zero generalized quadratic variation. By means of forward integrals a hedge against Markov–type claims is contructed. If Z is independent of W under some natural assumptions on Z and the admissible portfolio processes the model is shown to be arbitrage free. The fair price of the above claims appears to be the same as in the classical case $Z \equiv 0$. In particular, the Black–Scholes formula remains valid for non–semimartingale models with long range dependence.

0 Introduction

A bond and stock model with price processes

$$B(t) = B(0) \exp \left\{ \int_0^t r(s)\, ds \right\}$$

and

$$S(t) = S(0) \exp \left\{ \int_0^t \big(b(s) - \sigma^2/2\big) ds + \sigma \big(W(t) + Z(t)\big) \right\}$$

on the interval $[0, T]$ is considered. W is a Wiener process and Z is a continuous process with vanishing generalized quadratic variation. The interest rates $r(t)$ and the mean rates of return $b(t)$ are bounded measurable processes adapted to the filtration given by $Y := W + Z$. Related stochastic differential equations are understood in the sense of stochastic forward integrals as studied in [12] – [15].

The case $Z \equiv 0$ corresponds to the classical model which has been investigated within martingale theory. In particular, the well-known Black–Scholes formula for prices of options on the stock relies on martingale properties.
In the present paper a process Z is added which needs not be a semimartingale. This also permits long range dependence in the stock price development.
The corresponding portfolio valuing process X is introduced in Section 2 by means of self–financing strategies. If r is càglàd, a similar representation as known from semimartingale theory holds true (Proposition 2). For constant r in Theorem 1 a hedge against a terminal claim of the form $C = h(S(T))$ for a continuous function h with at most polynomial growth is constructed. It is determined by means of the same partial differential equation as used for the case $Z \equiv 0$. The essential tool is the Itô formula for the forward integral which holds also for non–semimartingale processes with generalized quadratic variations.
In order to treat the problem of arbitrage and pricing a claim the conditions on Z ind Section 4 are further restricted. It is supposed that Z is independent of W and has with probability 1 (w.p.1) fractional derivatives of order α in L_2 for some $\alpha > 1/2$. Balls of radius ε are now determined by the sum of the L_∞–norms of the trajectories of Z and of the L_2–norms of their fractional derivatives. We suppose that for every $\varepsilon > 0$ the ε–ball has positive probability. An example for Z is fractional Brownian motion with Hurst exponent $H > 1/2$.
For such integrators Z the forward integral has nice continuity properties. (The relationship between the type of stochastic integration and presence or absence of arbitrage is discussed in Section 4.) Under some natural conditions on the portfolio processes (fractional differentiability of order $1 - \alpha$ and a certain weak continuity w.r.t. the stock price development) we prove in Theorem 2 for deterministic rates $r(t)$ and $b(t)$ that there is no arbitrage opportunity. Moreover, Theorem 3 shows that the fair price of a claim $C = h(S(T))$ basing on strongly admissible portfolios of the above type is given by the corresponding value for $Z \equiv 0$ if r is constant. In particular, the classical Black–Scholes formula for European call options remains valid for our extended arbitrage free model with long range dependence. (The situation is similar to the case when working with equivalent risk neutral probability measures.)
In Theorem 3 it is also shown that the hedge process constructed in Theorem 1 under the additional assumptions on Z is strongly admissible in the above sense.

1 The bond and stock model

On a basic probability space $[\Omega, \mathcal{F}, \mathbb{P}]$ a **Wiener process W** and a (not necessarily independent) **continuous process Z** with **vanishing generalized quadratic process [Z]** and $Z(0) = 0$ are given. We consider a finite time interval $[0, T]$.

The above bracket is defined for continuous Z by

$$[Z](t) := \lim_{\substack{\varepsilon\to 0\\ (\text{ucp})}} \int_0^1 \varepsilon u^{\varepsilon-1} \, \frac{1}{u} \int_0^t \big(Z_t(s+u) - Z_t(s)\big)^2 \, ds \, du$$

if the limit exists, where (ucp) means uniform convergence in probability. (For a continuous function f we denote $f_t(s) := \big(f(s) - f(t)\big)\, 1_{[0,t]}(s),\ s \geq 0$.)
The **stochastic forward integral** of a measurable process Y with respect to (w.r.t.) continuous Z is given by

$$\int_0^t Y dZ := \lim_{\substack{\varepsilon\to 0\\ (\text{ucp})}} \int_0^1 \varepsilon u^{\varepsilon-1} \int_0^t Y(s) \, \frac{1}{u}\big(Z_t(s+u) - Z_t(s)\big) \, ds \, du$$

if the limit exists (Y is integrable w.r.t. Z).
These are slight extensions of the corresponding notions introduced by Russo and Vallois [6], [7]. Other construction of forward integrals, in particular, the Itô integral and the bracket of semimartingales are contained as special cases (see the discussion in [14]).
In the sequel we will need the following integration rule for the forward integral.

Proposition 1 *Let Y be integrable with respect to a continuous process Z and Ψ be a differentiable process with càglàd derivatives on $(0,T)$. Denote*

$$X(t) := \int_0^t Y dZ$$

Then ΨY is Z–integrable, Ψ is X–integrable and we have

$$\int_o^t \Psi \; dX = \int_0^t \Psi \; Y dZ.$$

Proof First note that continuity of Z and the definition of the forward integral imply continuity of X (cf. [13]). Further, it is easy to see that for C^1–integrands and continuous integrators the forward integral agrees pathwise with the Lebesgue–Stieltjes integral, i.e.,

$$\int_0^t \Psi \; dX = \Psi(t)\, X(t) - \int_0^t \Psi'(s)\, X(s)\, ds = \Psi(t)\, X(t) - \int_0^t \Psi'(s) \int_0^s Y dZ \; ds.$$

The order of integration in the last integral may be changed by the following arguments: In the definition of the forward integral we may work with subsequences converging with probability 1. Then the uniform convergence in s

of the Lebesgue integrals approximating the inner forward integral justifies the changes of the order of integration. Hence,

$$\begin{aligned}\int_0^t \Psi\ dX &= \Psi(t)\,X(t) - \int_0^t Y(v) \int_v^t \Psi'(s)\ ds\ dZ(v)\\ &= \Psi(t)\,X(t) - \int_0^t Y(v)\ \big(\Psi(t)-\Psi(v)\big)\ dZ(v)\\ &= \Psi(t)\,X(t) - \Psi(t)\int_0^t Y\ dZ + \int_0^t \Psi Y dZ\\ &= \int_0^t \Psi Y dZ.\end{aligned}$$

In particular, the calculations show that ΨY is Z–integrable. □

For Z as above the process $W+Z$ is now the driving process of our stock price development. Then we obtain from $[Z]\equiv 0$ and the Cauchy–Schwarz inequality, $[Z,W]\equiv 0$ (the generalized quadratic covariation is defined similarly as the quadratic variation) and therefore

$$[W+Z](t) = [W](t) = t.$$

Part of the results will even be derived for arbitrary continuous processes $\tilde{Z}$ with $[\tilde{Z}](t)=t$.
In order to get arbitrage free models we specify in Section 4 to processes Z independent of W with certain fractional smoothness properties and positive small ball probabilities. An example of such Z is **fractional Brownian motion $\mathbf{B^H}$** with Hurst exponent $H\in(1/2,1)$. Recall that this is a continuous Gauss process with mean zero and variances

$$\mathbb{E}\big(B^H(t)-B^H(s)\big)^2 = |t-s|^{2H}$$

starting at zero, which possesses **long range dependence**. For general processes Z with stationary increments the last property means that the series $\sum_{n=1}^{\infty} r(n)$ diverges, where

$$r(n) := \mathrm{Cov}\big(Z(1), Z(n+1)-Z(n)\big).$$

Let $(\mathcal{F}_t)_t\in[0,T]$ be the augmentation under $\mathbb{P}$ of the **filtration** given by the process $W+Z$.

We now consider a financial market with a **bond** and a **stock**, whose prices satisfy the stochastic differential equations (w.r.t. the forward integral)

$$d\,B(t) = r(t)\;B(t)\;dt \tag{1}$$
$$d\,S(t) = b(t)\;S(t)\;dt + \sigma\,S(t)\;d(W+Z)(t) \tag{2}$$

respectively. The **interest rates r(t)** and the **mean rates of return** $b(t)$ are measurable, bounded, $(\mathcal{F}_t)$–adapted processes and σ is a nonnegative constant.

$$\sigma(t) := \sigma\;\mathrm{Var}\big(W(t)+Z(t)\big)$$

is called **volatility** of the model.
The Itô formula for the forward integral (see [13], Theorem 5.5) implies that

$$S(t) = S(0)\exp\left\{\int_0^t \big(b(s)-1/2\sigma^2\big)\,ds + \sigma\big(W(t)+Z(t)\big)\right\} \tag{3}$$

is a pathwise solution to (2) with initial value $S(0)$. Uniqueness in the class of all processes admitting a generalized bracket and satisfying the stochastic Itô calculus follows from Theorem 7.1.1 in [13]. (For more general versions of (2) see [15].)

2 The portfolio valuing process

We now follow the portfolio valuing procedure as known from the case $Z \equiv 0$ (see, e.g., Karatzas and Shreve [3], Section 5.8): An investor starts with initial capital $x \geq 0$ and invests in continuous time $\varphi_0(t)$ shares in the bond and $\varphi(t)$ shares in the stock without consumption. (The assumptions on φ_0, φ_1 will be specified later.) Then the **value of the investment** at time t is

$$X(t) := \varphi_0(t)\,B(t) + \varphi_1(t)\,S(t)\,.$$

We consider a **self–financing strategy** whose discrete version reads

$$X(t+\Delta) - X(t) = \varphi_0(t)\big(B(t+\Delta)-B(t)\big) + \varphi(t)\big(S(t+\Delta)-S(t)\big)$$

i.e., there are no payoffs. According to (1) and (2) we use the following continuous version

$$dX(t) = \varphi_0(t)\,r(t)\,B(t)\,dt + \varphi(t)(b(t)\,S(t)\,dt + \sigma S(t)\,d(W+Z)(t)).$$

The shares $\varphi_0(t)$ and $\varphi(t)$ must be chosen such that the corresponding Lebesgue integrals and the stochastic forward integral exist. Denoting

$$\pi_0(t) := \varphi_0(t)\,B(t)$$
$$\pi(t) := \varphi(t)\,S(t)$$

we obtain

$$X(t) = \pi_0(t) + \pi(t)\,.$$

Then the above SDE may be rewritten as

$$dX(t) = r(t)\,X(t)\,dt + \big(b(t) - r(t)\,\pi(t)\,dt + \sigma\pi(t)\,d(W+Z)(t)\,.$$

Definition 1 *A* ***portfolio process*** $\boldsymbol{\pi(t)}$ *is a measurable adapted process for which the (forward) integrals corresponding to this SDE exist.*

In the sequel π is always a portfolio process. Denote

$$\tilde{W}(t) := W(t) + \frac{1}{\sigma}\int_0^t \big(b(s) - r(s)\big)\,ds \tag{4}$$

In these terms the last equation reads

$$dX(t) = r(t)\,X(t)\,dt + \sigma\pi(t)\,d(\tilde{W}+Z)(t) \tag{5}$$

and the forward integral $\int_0^t \pi\,d(\tilde{W}+Z)$ exists.
Note, that until now we have not used that W is the Wiener process. If $r \equiv b$ then we get $\tilde{W} = W$. Assume that r and b are deterministic functions and Z is independent of the Wiener process W. Then the Girsanov transformation (see, e.g. [3], p. 374) leads to a probability distribution $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ under which $\tilde{W}$ is also a Wiener process independent of Z.
Turning back to (5) we obtain the usual representation of the portfolio value process X by the following (for the special case $\tilde{Z} = \tilde{W} + Z$).

Proposition 2 *Suppose r is càglàd $\tilde{Z}$ is a continuous process and π is integrable w.r.t $\tilde{Z}$ in the sense of the forward integral. Then the unique continuous solution X to the SDE*

$$dX(t) = r(t)\,X(t)\,dt + \sigma\pi(t)\,d\tilde{Z}(t)$$

with initial value $X(0) = x$ is given by

$$X(t) = e^{\int_0^t r(s)\,ds}\left(x + \int_0^t e^{-\int_0^s r(u)\,du}\sigma\pi(s)\,d\tilde{Z}(s)\right)\,. \tag{6}$$

Proof According to Proposition 1 the forward integral in (6) exists. Moreover, the proof of 1 implies for $\Psi(t) := e^{\int_0^t r(s)\,ds}$, $Y(t) := \Psi(t)^{-1}\sigma\pi(t)$ and

X from (6)

$$\int_0^t r(s)\,X(s)\,ds = \int_0^t \Psi'(s)\left(x + \int_0^s Y\,d\tilde{Z}\right)\,ds$$
$$= \Psi(t)\int_0^t Y\,d\tilde{Z} - \int_0^t \Psi\,Y\,d\tilde{Z} + x\big(\Psi(t)-1\big)$$
$$= X(t) - x - \int_0^t \sigma\pi\,d\tilde{Z}\,,$$

i.e., this X is a solution of the SDE. Uniqueness follows from the local contraction principle pathwise applied. □

Below we again specify to the case $\tilde{Z} := \tilde{W} + Z$ and adapted π.

Definition 2 *The portfolio process π of the self–financing strategy is said to be **admissible** for the initial value $X(0) = x \geq 0$ if $X(t) \geq 0,\ t \in (0,T],\ \mathbb{P}-$a.s.*

3 Hedging a claim

Definition 3 *A terminal **claim** on the stock is a nonnegative $\mathcal{F}_T$–measurable random variable C (interpreted as a payoff at maturity T).*

Standard examples 1

(i) **European call option:**

$$C := \big(S(T) - K\big)_+$$

(ii) **European put option:**

$$C := \big(K - S(T)\big)_+$$

(In both cases K is the exercise price of the option.)

(iii) **Down–and–out barrier call option**

$$C := \big(S(T) - K\big)_+ 1_{\{S(t)>v,\, t\in[0,T]\}}$$

for some barrier value v.

(iv) **Asian option:**

$$C := \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+$$

(v) **Lookback option:**

$$C := S(T) - \inf_{t\in[0,T]} S(t)\,.$$

(The last three types are called exotic options.)

Definition 4 *The claim C is said to be* ***replicable*** *if there exists an admissible portfolio process π such that for the wealth process X given by (6) we have*

$$X(T) = C \quad \mathbb{P} - a.s.$$

In this case X is called ***hedge process*** *against the claim C.*

Definition 5 *The* ***fair price*** *of the claim C at $t = 0$ is given by*

$c_* := \inf\{x \geq 0 :$ *there exists a hedge process X against C with* $X(0) = x\}$.

The following theorem shows that Markov–type claims may be hedged in the same way as for the case of the Wiener process if the driving process $\tilde{Z}$ has the same quadratic variation.

Theorem 1 *Let $\tilde{Z}$ be a continuous process with $[\tilde{Z}](t) = t$, r, σ be positive constants and define the process*

$$S(t) := S(0)\, \exp\{(r - \sigma^2/2)t + \sigma\, \tilde{Z}(t)\}.$$

For a continuous function $h : (0, +\infty) \to [0, +\infty)$ with at most polynomial growth the function $c(s,t)$ on $[0, +\infty) \times [0, +\infty)$ is determined as the unique solution of the Cauchy problem

$$-\frac{\partial c}{\partial t} + rs\,\frac{\partial c}{\partial s} + \frac{1}{2}\,\sigma^2\, s^2\, \frac{\partial^2 c}{\partial s^2} - rc = 0, \qquad c(s,0) = h(s)\,.$$

Then the process

$$X(t) := c\big(S(t), T - t\big), \quad t \in [0,T]$$

is representable as

$$X(t) = e^{rt}\left(x + \int_0^t e^{-rs}\ \sigma\pi(s)\ d\tilde{Z}(s)\right)$$

with

$$\pi(t) := \frac{\partial c}{\partial s}\ \big(S(t), T - t\big)$$

and

$$X(T) = h\big(S(T)\big)\,.$$

Remark 1 From the case $\tilde{Z} = W$ it is well–known that

$$c(s,t) = e^{-rt}\,\mathbb{E}\ h\Big(s\ \exp\big\{(r-\sigma^2/2)\,t + \sigma W(t)\big\}\Big).$$

For $h(s) = (s-K)_+$ an explicit expression is given by the **Black–Scholes formula** (see Black and Scholes [1] or the monographs Karatzas and Shreve [3], [4], and Shiryaev [10]).

Proof of Theorem 1. (See also Schoenmakers and Kloeden [9] for the special case where $\tilde{Z} = W + Z$, $W$is the Wiener process, Z a continuous process independent of W and $[Z] \equiv 0$.)
The process $\tilde{Z}$ satisfies the Itô formula w.r.t. the smooth transformation

$$c\big(S(t), T-t\big) = c\Big(S(0)\ \exp\big\{(r-\sigma 2/2)\,t + \sigma\tilde{Z}(t)\big\}, T-t\Big)$$

([13], Theorem 5.5). In particular, this includes existence of the corresponding stochastic forward integral. Using the partial differential equation for c on infers that $X(t)$ satisfies the SDE in Proposition (2) with

$$\pi(t) = \frac{\partial c}{\partial s}\ \big(S(t), T-t\big).$$

(The integrand in the time integral of the Itô formula vanishes.) Then (6) yields the asserted representation of X. By construction,

$$X(T) = c\big(S(T), 0\big) = h\big(S(T)\big).$$

□

4 Strong arbitrage and pricing a claim

We now turn back to the case $\tilde{Z} = \tilde{W} + Z$ with $\tilde{W}$ given by (4). Definition 5 of the fair price c_* of a claim C and Theorem 1 imply the following for constant interest rate r and $C = h\big(S(T)\big)$:

Corollary 1

$$c_* \le c\big(S(0), T\big) = \mathbb{E}\ h\Big(S(0)\ \exp\big\{(r-\sigma^2/2)\,T + \sigma W(T)\big\}\Big)$$

and for the initial value $x := c\big(S(0), T\big)$ *there is a hedge process* X *as above.*

In order to get equality in this estimate we formulate some natural conditions on the process Z and the admissible portfolios π. First recall the classical case. If $\tilde{Z} = \tilde{W} + Z$ is a semimartingale equivalent to the Wiener process then the fair price is given by the above expectation (see [4], 5.8). Moreover, the model is arbitrage free, which can easily be seen from the representation (6) for $Z \equiv 0$ working with the equivalent distribution. Recently it has been shown by M. Ndoye (personal communication) that for fractional Brownian motion B^H independent of W the process

$$\tilde{Z} = W + B^H$$

is a semimartingale equivalent to a Wiener process if $H \in (3/4,\, 1)$ and for $H \in (1/2,\, 3/4)$ it is no semimartingale. Below the case $H \in (1/2,\, 1)$ is included.

In general, the notion of arbitrage reads as follows.

Definition 6 *The bond and stock model as above admits an* ***arbitrage*** *if there exists a portfolio which is admissible for the initial value $x = 0$, i.e. $X(t) \geq 0,\ t \in [0,T]$, w.p.1, such that $\mathbb{P}\big(X(T) > 0\big) > 0$.*

Recall that the presence or absence of arbitrage depends very much on the driving process $\tilde{Z}$ (in our case $\tilde{Z} = \tilde{W} + Z$) and on the type of stochastic integrals that are used. If one takes $\tilde{Z} = B^H,\ H \in (1/2,\, 1)$, and the forward integral then there exists an arbitrage opportunity (cf. Rogers [5] and Shiryaev [10], VII 2c, 4, for a simpler construction). Hu and Øksendal [2] have shown that for B^H as above and the stochastic integral defined by Wick products there is no arbitrage. In the present paper we prefer the forward integral, since it agrees for step functions with the Riemann–Stieltjes sums. Under some additional assumptions on Z and π one can work with approximation by such sums which leads to fast computer procedures. Moreover, under additional conditions the choice $\tilde{Z} = \tilde{W} + Z$ enables us to transfer the problem of fair prices and no arbitrage to the case of the Wiener process. This provides a certain analogy to the semimartingale case.

Conditions on Z:

Consider the following **Liouville-type spaces** of fractional order on $[0,T]$. For $1/2 < \alpha < 1$ let $\mathfrak{I}^{\alpha}_{T-}(L_2)$ be the set of those L_2–functions f for which the fractional derivatives

$$D^{\alpha}_{T-} f_T(t) := \frac{1}{\Gamma(\alpha)} \left(\frac{f(t) - f(T)}{(T-t)^{\alpha}} + \alpha \int_t^T \frac{f(s) - f(t)}{(s-t)^{\alpha+1}}\, ds \right)$$

are defined in the L_2–sense. $I^{1-\alpha}_{0+}(L_2)$ is given similarly, where instead of right–sided derivatives of f_T the left–sided versions of f are chosen:

$$D^{1-\alpha}_{0+} f(t) := \frac{1}{\Gamma(\alpha))} \left(\frac{f(t)}{t^{\alpha}} + \alpha \int_0^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}}\, ds \right).$$

(see Samko, Kilbas and Marichev [8]).

Remark. Instead of f we have chosen in the first definition the boundary corrected version f_T in order to avoid $f(T) = 0$.) The norm

$$\|f\|^{\alpha}_{2,T} := \|f\|_{L_\infty} + \|D^{\alpha}_{T-}\, f_T\|_{L_2}$$

in $\mathfrak{I}^{\alpha}_{T-}(L_2)$ will be used in the sequel. This space is embedded in the Hölder space of order $\alpha - 1/2$. The functions of $I^{1-\alpha}_{0+}(L_2)$ need not be continuous or bounded, in particular all finite steps functions are included.

In these terms the first condition of Z is formulated:

For some $\alpha \in (1/2, 1)$ the process Z is an element of $\mathfrak{I}^{\alpha}_{T-}(L_2)$ w.p.1 and for any $\varepsilon > 0$ we have

$$\mathbb{P}\left(\|Z\|^{\alpha}_{2,T} < \varepsilon\right) > 0. \tag{Z1}$$

Fractional Brownian motion B^H with $H \in (1/2, 1)$ provides an example of such a Z. This follows from Theorem 1.4 in Stolz [11].

We now additionally assume:

Z is independent of the Wiener process W. (Z2)

Conditions on π:
In order to describe the class of "strongly" admissible portfolios we now will work with a canonical representation of the basic probability space. There are measurable subsets (w.r.t. the Kolmogorov σ–algebra) Ω_1 and Ω_2 of the space of continuous functions such that

$$\mathbb{P}(W \in \Omega^1) = \mathbb{P}(Z \in \Omega^2) = 1.$$

We choose $\Omega = \Omega^1 \times \Omega^2$ as basic space with elements (ω^1, ω^2). In this case $\mathbb{P}$ is the product measure of the Wiener measure $\mathbb{P}^1$ and the distribution $\mathbb{P}^2$ of Z on the trace σ–algebras $\mathcal{F}^1$ and $\mathcal{F}^2$. The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is determined as before by $W + Z$. Let $(\mathcal{F}^1_t)_{t\in[0,T]}$ be the augmentation of the filtration induced by $\mathcal{F}^1$ w.r.t. $\mathbb{P}^1$. The first condition on the portfolios π reads now as follows:

(i) $0 < \int_0^T \pi^2(t)\,dt < \infty \quad , \quad \mathbb{P}$–a.s.

(ii) There is a subset $\tilde{\Omega}^2$ of Ω^2 with $\mathbb{P}^2(\tilde{\Omega}^2) = 1$ such that for any sequence $\omega^2_{(n)}$ from $\tilde{\Omega}^2$ with $\lim_{n\to\infty} \|\omega^2_{(n)}\|^{\alpha}_{2,T} = 0$ we have

$$\lim_{n\to\infty} \int_0^T \left(\pi(t, \omega^1, \omega^2_{(n)}) - \pi^1(t, \omega^1)\right)^2 dt = 0 \tag{Π1}$$

for convergence in probability w.r.t. to the Wiener measure $\mathbb{P}^1$ and some $(\mathcal{F}^1_T)$–adapted process π^1 which is non–zero in L_2 with positive probability.

This is a **weak stochastic continuity property** of the portfolio w.r.t. to the trajectories of $W + Z$ when Z becomes small.

In [13] we have shown that the forward integral $\int_0^t A \, dZ$ may be calculated by means of fractional derivatives provided that the summed order of differentiability of A and Z equals 1. This will be applied to the integrals

$$\int_0^t \bar{\pi} \, dZ$$

from (6) where

$$\bar{\pi}(t) = e^{-\int_0^t r(s)\, ds} \, \sigma \, \pi(t)$$

is the discounted portfolio process. Therefore the second condition on π is the **existence of fractional derivatives** of $\bar{\pi}$ of order $1-\alpha$ and a certain **weak boundedness** property of them:

(i) $\bar{\pi} \in I_{0+}^{1-\alpha}(L_2)$ $\mathbb{P}$–a.s., where $\alpha \in (1/2, 1)$ is from condition (Z1).

(ii) $\limsup_{n\to\infty} \int_0^T \Big(D_{0+}^{1-\alpha} \, \bar{\pi}\big(\cdot, \omega^1, \omega^2_{(n)}\big)(t)\Big)^2 \, dt < \infty$ $\qquad (\Pi 2)$

for $\mathbb{P}^1$–a.a. ω^1 and any sequence $\omega^2_{(n)}$ from a set of full $\mathbb{P}^2$–measure such that $\lim_{n\to\infty} \|\omega^2_{(n)}\|^{\alpha}_{2,T} = 0$.

For practical purposes in the sense of numerical discretization these conditions are not very restrictive: Since $1-\alpha < 1/2$ the space $I_{0+}^{1-\alpha}(L_2)$ has a sufficiently rich structure (cf. Remark 1).

Definition 7 *An admissible portfolio π is called* ***strongly admissible*** *for the driving process $W+Z$ with Z satisfying (Z1) and (Z2) if the conditions (Π1) and (Π2) are fulfilled.*

We now slightly specify the notion of arbitrage (cf. 6) under the conditions(Z1) and (Z2):

Definition 8 *The bond and stock model admits a* ***strong arbitrage*** *if there exists a strongly admissible portfolio π for the initial wealth $x = 0$ such that $\mathbb{P}\big(X(T) > 0\big) > 0$, where X is given by (6).*

The first main result concerns the **absence of strong arbitrage**:

Theorem 2 *Let W be the Wiener process and Z be a continuous process with $[Z] \equiv 0$ satisfying (Z1) and (Z2). The stock price is given by the equation (2), i.e.,*

$$d\,S(t) = b(t)\,S(t)\,dt + \sigma\,S(t)\,d(W+Z)(t)\,.$$

The interest rates $r(t)$ of the bond and the mean rates of return $b(t)$ of the stock are deterministic bounded functions, being càglàd and measurable, respectively, and $\sigma > 0$.
Then there is no strong arbitrage in the model.

Proof We show that for any strongly admissible portfolio π as above the associated process $\tilde{\pi}$ from $(\Pi 1)$ is an admissible portfolio for the driving process $\tilde{W}$ given by (4) with the same initial wealth x. From this we derive that a strong arbitrage opportunity does not exist.
For, we work with the equivalent measure $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}^1 \times \mathbb{P}^2$ (cf. the arguments before Proposition 2), where $\tilde{W}$ is a Wiener process w.r.t. $\tilde{\mathbb{P}}^1$. Since $\bar{\pi}$ is adapted to the augmented filtration induced by $\tilde{W} + Z$ and $\tilde{W}$ and Z are independent we obtain that the first summand in the representation

$$\int_0^t \bar{\pi}\, d(\tilde{W} + Z) = \int_0^t \bar{\pi}\, d\tilde{W} + \int_0^t \bar{\pi}\, dZ$$

agrees with the **Itô integral**. Furthermore, $Z \in \mathfrak{I}^{\alpha}_{T-}(L_2)$ and $\bar{\pi} \in I^{1-\alpha}_{0+}(L_2)$ a.s., and therefore the second forward integral may be calculated pathwise by means of fractional derivatives (see [13]):

$$\int_0^t \bar{\pi}\, dZ = \int_0^T D^{1-\alpha}_{0+}\,(1_{(0,t)}\bar{\pi})(s)\, D^{\alpha}_{T-}\, Z_T(s)\, ds$$

by the conditions (Z1), (i) and $(\Pi 2)$,(i). In view of

$$\sup_{t\in[0,T]} \int_0^T \left(D^{1-\alpha}_{0+}(1_{(0,t)}\bar{\pi})(s)\right)^2 ds \le \text{const} \int_0^T \left(D^{1-\alpha}_{0+}\bar{\pi}(s)\right)^2 ds$$

(see [8], Theorem 11.6), one obtains from this representation by means of the Cauchy–Schwarz inequality

$$\sup_{t\in[0,T]} \left|\int_0^t \bar{\pi}\, dZ\right|^2 \le \text{const} \int_0^T \left(D^{1-\alpha}_{0+}\, \bar{\pi}(s)\right)^2 ds \int_0^T \left(D^{\alpha}_{T-}\, Z_T(s)\right)^2 ds \qquad (7)$$

$\tilde{\mathbb{P}}$–a.s.
The idea is now as follows: Let π be admissible for $x \ge 0$, i.e.,

$$0 \le X(t)\, e^{\int_0^t r(s)\, ds} = x + \int_0^t \bar{\pi}\, d\tilde{W} + \int_0^t \bar{\pi}\, dZ\,, \qquad (8)$$

$0 \le t \le T, \quad \tilde{\mathbb{P}}$–a.s.
By the small ball property (Z1),(ii) we can select for each $n \in \mathbb{N}$ an element $\omega^2_{(n)}$ from a set of full $\mathbb{P}^2$–measure such that

$$\|\omega^2_{(n)}\|^{\alpha}_{2,T} < \frac{1}{n}$$

and the inequalities (7) and (8) hold true for $(\omega^1, \omega^2_{(n)})$ for any $n \in \mathbb{N}$ and $\tilde{\mathbb{P}}^1$–almost all $\omega^1 \in \Omega^1$.
Moreover, by condition $(\Pi 1)$,(ii) the sequence $\omega^2_{(n)}$ may be chosen such that the integrand $\bar{\pi}(t, \omega^1, \omega^2_{(n)})$ converges in probability w.r.t. $\tilde{\mathbb{P}}^1$ and in L_2 w.r.t. the parameter t to some $(\mathcal{F}^1_t)$–adapted process

$$\bar{\pi}^1(t, \omega^1) := e^{-\int_0^t r(s)ds} \sigma\, \pi^1(t, \omega^1)\,.$$

Then the Itô integral in (8)

$$\int_0^t \bar{\pi}(s, \omega^1, \omega^2_{(n)})\, d\tilde{W}(\omega^1, s)$$

converges uniformly in probability to

$$\int_0^t \bar{\pi}^1(s, \omega^1)\, d\tilde{W}(\omega^1, s)$$

as $n \to \infty$. The second integral in (8) according to (7) may be estimated by

$$\text{const} \int_0^T \left(D^{1-\alpha}_{0+}\, \bar{\pi}(\cdot, \omega^1, \omega^2_{(n)})(s)\right)^2 ds\ \left(\|\omega^2_{(n)}\|^{\alpha}_{2,T}\right)^2 .$$

This tends to zero as $n \to \infty$ for $\tilde{\mathbb{P}}^1$–a.a. ω^1, since the last integrals may assumed to be bounded in view of $(\Pi 2)$,(ii). Taking these limits in (8) we infer

$$0 \le x + \int_0^t \bar{\pi}\, d\tilde{W}\,,$$

$0 \le t \le T$, $\tilde{\mathbb{P}}^1$–a.s. For $x = 0$ this is impossible unless $\bar{\pi}^1$ vanishes in L_2 $\tilde{\mathbb{P}}^1$–a.s. by the properties of the Itô integral (see [3], [10]). In $(\Pi 1)$,(ii) we have supposed that π^1 does not vanish with positive probability. Hence, there is no strongly admissible portfolio π for $x = 0$. □

Similarly as for the notion of arbitrage we now adapt the hedges to our situation, i.e., to the conditions (Z1) and (Z2).

Definition 9 *The process X is called a* ***strong hedge*** *against the claim C if it is a hedge in the sense of Definition 4 with a strongly admissible portfolio process π.*

The following notion of the price of a claim is related to type of stochastic integrals we use under conditions (Z1) and (Z2).

Definition 10 *The **price** of a claim C at $t = 0$ is given by*

$$c^* := \inf\{x : \text{there exists a strong hedge process } X \text{ against } C \text{ with } X(0) = x\}.$$

For claims of the form $C = h\big(S(T)\big)$ with a continuous nonnegative function h with at most polynomial growth the value c^* may be determined explicitly:

Theorem 3 *Suppose that the bond and stock model is as in the Arbitrage theorem 2 with constant interest rate r. Let the function $c(s,t)$ be determined by the Cauchy problem in Theorem 1 with initial value $c(s,0) = h(s)$. Then the price of the claim $C = h(S(T))$ is given by*

$$c^* = c\big(S(0), T\big).$$

In particular, for European call options the classical Black–Scholes formula remains valid.

Proof Suppose that

$$\begin{aligned} X(t) =& e^{rt}\left(x + \int_0^t \bar{\pi}\, d\big(\tilde{W} + Z\big)\right), \quad 0 \le t \le T, \\ X(T) =& h\big(S(T)\big) \end{aligned}$$

is a strong hedge as above against C.
From the proof of Theorem 2 it follows that for some $\bar{\pi}^1$ adapted to the Wiener process $\tilde{W}$ we get

$$0 \le X^1(t) := e^{rt}\left(x + \int_0^t \bar{\pi}^1\, d\tilde{W}\right), \quad 0 \le t \le T$$

$\tilde{\mathbb{P}}^1$–a.s. Moreover, the value $X^1(T,\omega^1)$ arises as the limit in $\tilde{\mathbb{P}}^1$–probability as $n \to \infty$ of $X(T,\omega^1,\omega^2_{(n)})$ for some sequence $\omega^2_{(n)}$ of continuous functions converging in L_∞ to zero. In particular,

$$\lim_{n\to\infty} \omega^2_{(n)}\,(T) = 0\,.$$

On the other hand, the sequence may be chosen so that

$$X(T,\omega^1,\omega^2_{(n)}) = h\big(S(t,\omega^1,\omega^2_{(n)})\big)$$

for $\tilde{\mathbb{P}}^1$–a.a. ω^1 and $n \in \mathbb{N}$. Since h is continuous and

$$S(T,\omega^1,\omega^2_{(n)}) = S(0)\ \exp\left\{\int_0^T \big(b(t) - \sigma^2/2\big)dt + \sigma\big(\omega^1(T) + \omega^2_{(n)}(T)\big)\right\},$$

the right hand side tends to $h\big(S(T,\omega^1,0)\big)$as $n\to\infty$, hence

$$X^1(T,\omega^1)=h\big(S(T,\omega^1,0)\big).$$

This means that the process X^1 is a hedge against the claim C for the model with the Wiener process where $Z\equiv 0$. Then we are in the situation of martingale theory for pricing a claim. It follows (see [3], [4], [10]) that

$$x=c\big(S(0),T\big)$$

for the initial value x of any strong hedge process X.
It remains to show existence of a strong hedge. For, we use the portfolio process

$$\pi(t):=\frac{\partial c}{\partial s}\,\big(S(t),T-t\big)$$

from Theorem 1 generating a hedge process X and prove that π satisfies the conditions $(\Pi 1)$ and $(\Pi 2)$.

Since $\dfrac{\partial c}{\partial s}$ is continuous, it follows from the above representation of $S(t,\omega^1,\omega^2)$ that the portfolio π^1 in the condition $(\Pi 1)$,(ii) is given by

$$\pi^1(t,\omega^1)=\frac{\partial c}{\partial s}\,\big(S(t,\omega^1,0),T-t\big)\,.$$

Furthermore,

$$\bar\pi(t)=e^{-rt}\;\frac{\partial c}{\partial s}\;\Big(S(0)\,\exp\big\{(r-\sigma^2/2)\,t+\sigma(\tilde W(t)+Z(t))\big\},T-t\Big)\,.$$

By the smoothness of the exponential function and that of $\dfrac{\partial c}{\partial s}$ the problem whether $\bar\pi\in I_{0+}^{1-\alpha}(L_2)$ $\mathbb{P}$–a.s. may be reduced to that for $\tilde W+Z$ w.r.t. $\tilde{\mathbb{P}}$. (Use the mean value theorem in the integral version. Cf. [13].) The Wiener process has a.s. sample paths of Hölder continuity of all orders less than 1/2. Therefore $\tilde W\in I_{0+}^{1-\alpha}(L_2)$, $\tilde{\mathbb{P}}^1$–a.s. For the process Z we have supposed $Z\in\mathfrak{I}_{T-}^{\alpha}(L_2)$, $\mathbb{P}^2$–a.s. Since $1-\alpha<1/2<\alpha$ this implies $Z\in I_{0+}^{1-\alpha}(L_2)$ w.p.1. Moreover,

$$\int_0^T\big(D_{0+}^{1-\alpha}\,Z_T(t)\big)^2\,dt\le\text{const}\;\int_0^T\big(D_{T-}^{\alpha}\,Z_T(t)\big)^2\,dt$$

(see [8], Corollary 1 of Theorem 11.5). In particular, $(\Pi 2)$,(i) is proved.
Similarly, the condition $(\Pi 2)$,(ii) on $\bar\pi$ may be reduced to that on $\tilde W+Z$. Thus, it is enough to show that

$$\left(\int_0^T\big(D_{0+}^{1-\alpha}(\omega^1+\omega^2_{(n)})(t)\big)^2\,dt\right)^{1/2}$$

for $\mathbb{P}^1$–a.a. ω^1 is bounded by some $C(\omega^1)$ if $\lim_{n\to\infty} \|\omega^2_{(n)}\|^\alpha_{2,T} = 0$. This expression does not exceed

$$\left(\int_0^T \left(D_{0+}^{1-\alpha}\,\omega^1(t)\right)^2 dt\right)^{1/2} + \left(\int_0^T \left(D_{0+}^{1-\alpha}\,\omega^2_{(n)}(t)\right)^2 dt\right)^{1/2}.$$

The first summand is finite $\mathbb{P}^1$–a.s. The second one agrees with

$$\left(\int_0^T \left(D_{0+}^{1-\alpha}(\omega^2_{(n)})_T(t) + D_{0+}^{1-\alpha}(\omega^2_{(n)}(T)\,1_{[0,T)})(t)\right)^2 dt\right)^{1/2}$$

$$\leq \left(\int_0^T \left(D_{0+}^{1-\alpha}\,(\omega^2_{(n)})_T\,(t)\right)^2 dt\right)^{1/2} + \text{ const } \omega^2_{(n)}(T).$$

in view of the above inequality for the last integral (in terms of Z) this sum may be estimated by const $\|\omega^2_{(n)}\|^\alpha_{2,T}$ which tends to zero by assumption. Consequently, $(\Pi 2)$,(ii) is also fulfilled. □

References

1. Black, F. and Scholes, M.: The pricing of options and corporate liabilities, J. Political Economics **81**, (1973) 637-654.
2. Hu, Y. and Øksendal, B.: Fractional white noise calculus and applications to finance
3. Karatzas, I. and Shreve, S. E.: *Brownian Motion and Stochastic Calculus*, Springer 1988.
4. Karatzas, I. and Shreve, S. E.: *Methods of Mathematical Finance*, Springer 1998.
5. Rogers, L. C.: Arbitrage with fractional Brownian motion, Math. Finance **7** (1997), 95–105 .
6. Russo, F. and Vallois, P.: Forward, backward and symmetric stochastic integration, Probab. Th. Relat. Fields **97** (1993), 503–421.
7. Russo, F. and Vallois, P.: The generalized covariation process and Itô formula, Stoch. Processes Appl. **59** (1995), 81–104.
8. Samko, S. G., Kilbas, A. A. and Marichev, O. I.: *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach 1993
9. Schoenmakers, J.G. M. and Kloeden, P: Robust option replication for a Black–Scholes model extended with nondeterministic trends, J. Appl. Math. Stoch. Anal. **12** (1999), 113–120.
10. Shiryaev, A. N.: *Essentials of Stochastic Finance. Fact, Models, Theory*, World Scientific 1999.
11. Stolz, W.: Small ball probabilities for Gaussian processes under uniform norms, J. Theoret. Probab. **9** (1996), 613-630.
12. Zähle, M.: Integration with respect to fractal functions and stochastic calculus I, Probab. Th. Relat. Fields **111** (1998), 333–374.

13. Zähle, M.: Integration with respect to fractal functions and stochastic calculus II, Math. Nachr. **225** (2001), 145–183.
14. Zähle, M.: Forward integrals and stochastic differential equations. In: Progress in Probab., Birkhäuser.
15. Zähle, M.: Pathwise solution of anticipating stochastic differential equations, (Preprint).