Forward Integrals and Stochastic Differential Equations

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Abstract. We show that an anticipating stochastic forward integral introduced in [8] by means of fractional calculus is an extension of other forward integrals known from the literature. The latter provide important classes of integrable processes. In particular, we investigate the deterministic case for integrands and integrators from optimal Besov spaces. Here the forward integral agrees with the continuous extension of the Lebesgue–Stieltjes integral to these function spaces.

Generalized quadratic variation processes are defined in a similar manner. A survey on applications to anticipating stochastic differential equations with driving processes $Z^0, Z^1, \ldots, Z^m$ is given, where Z^0 has a generalized bracket $[Z^0]$ and $[Z^0], Z^1, \ldots, Z^m$ are smooth of fractional order greater than $1/2$.

1. Anticipating stochastic differential equations

We consider SDE in $\mathbb{R}^n$ of the following type:

$$(1) \qquad dX(t) = \sum_{i=0}^{m} a_i\big(X(t),t\big)\, dZ^i(t) + b\big(X(t),t\big)\, dt\ ,\ \ t \in [0,T]\ ,\ \ X(0) = X_0\,.$$

$a_0(x,t), \ldots, a_m(x,t)$ are random C^1–vector fields whose partial derivatives are locally Lipschitz in x and $b(x,t)$ is continuous and locally Lipschitz in x. X_0 is an arbitrary random vector and the driving processes are as follows: Z^0 is continuous and possesses a generalized bracket $[Z^0](t)$ (see section 2) and $[Z^0], Z^1, \ldots, Z^m$ have sample paths in the Sobolev–Slobodeckij space W_2^β (which agrees with the Besov space $B_{2,2}^\beta$) for some $\beta > 1/2$. We do not suppose any adaptedness or independence. The stochastic integrals in (1) are forward integrals and will be introduced in section 2. An important example with applications in mathematical finance is the case where Z^0 agrees with the Wiener process and $Z^1, \ldots, Z^m$ are fractional Brownian motions with Hurst exponents greater than $1/2$.

In [9] it is proved that equation (1) has a unique local pathwise solution with generalized bracket satisfying the rules of Itô calculus. It may be determined by the following procedure (which extends the Doss–Sussman approach known from martingale theory for $m = 0$):

$$X(t) = h\big(Y(t), Z^0(t), t\big) \tag{2}$$

where $h(y,z,t)$ is a pathwise local C^1–solution of the differential equation

$$\begin{aligned} \frac{\partial h}{\partial z}\,(y,z,t) &= a_0\big(h(y,z,t),t\big) \\ h\big(Y(0),Z(0),0\big) &= X_0 \end{aligned} \tag{3}$$

being Lipschitz in y such that

$$\det\left(\frac{\partial h}{\partial y}\,(y,z,t)\right) \neq 0.$$

Given $Y(0)$ and h the stochastic process Y in $\mathbb{R}^n$ is locally determined by the SDE

$$\begin{aligned} dY(t) = \left(\frac{\partial h}{\partial y}\,\big(Y(t),Z^0(t),t\big)\right)^{-1} &\Bigg[\sum_{j=1}^{m} a_j\Big(h\big(Y(t),Z^0(t),t\big),t\Big)\,dZ^j(t) \\ &+ \bigg(b\Big(h\big(Y(t),Z^0(t),t\big),t\Big) - \frac{\partial h}{\partial t}\,\big(Y(t),Z^0(t),t\big)\bigg)\,dt \\ &- \frac{1}{2}\,\frac{\partial a_0}{\partial x}\,\Big(h\big(Y(t),Z^0(t),t\big),t\Big)\,a_0\Big(h\big(Y(t),z^0(t),t\big),t\Big)\,d[Z^0](t)\Bigg]. \end{aligned} \tag{4}$$

This equation has a better analytical behavior than (1), in particular since the processes $Z^1,\ldots,Z^m$ have zero quadratic variations. In [9] a solution procedure by means of a contraction principle in Besov spaces is demonstrated.

Note that because of equation (3) the representation (2) of the unique solution $X(t)$ is, in general, not unique.

2. Stochastic forward integrals and generalized quadratic variations

By means of fractional calculus we are led in [8] to the following version of **stochastic forward integral**: Let X be a càglàd process and Z a càdlàg process on $[0,T]$. Denote $Z_{t-}(s) := \big(Z(s) - Z(t-)\big)\,1_{[o,t)}(s)$.

Definition. *X is integrable w.r.t Z on $[0,T]$ if the following limit exists (uniformly in t) in probability:*

$$\int_0^{t-} X\,dZ := \lim_{\varepsilon\to 0} \int_0^1 \varepsilon u^{\varepsilon-1} \int_0^t X(s)\,\frac{Z_{t-}(s+u) - Z_{t-}(s)}{u}\,ds\,du\,. \tag{5}$$

This is an **extension** of the forward integral

$$\lim_{u \searrow 0} \int_0^t X(s)\ \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u}\ ds \tag{6}$$

with (uniform) convergence in probability introduced in Russo and Vallois [4] for the continuous case (and in a modified version for the discontinuous case).

In order to check this note first that the kernel $\varepsilon u^{\varepsilon-1}$ acts as the δ–function as $\varepsilon \to 0$. Further, let $Y_t(u)$ be a family of càdlàg processes on (0,1) converging to $Y_t(0)$ in probability as $u \searrow 0$ (uniformly in the parameter t). Suppose that $\sup_{(t),u} Y_t(u) < \infty$ w.p.1. Then we have

$$\begin{aligned}
\int_0^1 \varepsilon u^{\varepsilon-1}\ Y_t(u)\ du &= Y(0) + \int_0^1 \varepsilon u^{\varepsilon-1}\ \big(Y_t(u) - Y_t(0)\big)\ du \\
&= Y(0) + \int_0^1 \varepsilon u^{\varepsilon-1}\ 1\big\{u : |Y_t(u) - Y_t(0)| > \delta\big\}\big(Y_t(u) - Y_t(0)\big)\ du \\
&\quad + \int_0^1 \varepsilon u^{\varepsilon-1}\ 1\big\{u : |Y_t(u) - Y_t(0)| \le \delta\big\}\big(Y_t(u) - Y_t(0)\big)\ du\,.
\end{aligned}$$

The last integral does not exceed δ. The first integral on the right–hand side converges to zero in probability as $\varepsilon \to 0$ (uniformly in t), since $|Y_t(u) - Y_t(0)|$ is (uniformly) bounded w.p.1 and

$$\begin{aligned}
&\lim_{\varepsilon \to 0}\ \big(\sup_t\big)\ \mathbb{E} \int_0^1 \varepsilon u^{\varepsilon-1}\ 1\Big\{u : \big|Y_t(u) - Y_t(0)\big| > \delta\Big\}\ du \\
&\le \lim_{\varepsilon \to 0} \int_0^1 \varepsilon u^{\varepsilon-1}\ \mathbb{P}\Big\{\big(\sup_t\big)\big|Y_t(u) - Y_t(0)\big| > \delta\Big\}\ du \\
&= \lim_{u \searrow 0}\ \mathbb{P}\Big\{\big(\sup_t\big)\big|Y_t(u) - Y_t(0)\big| > \delta\Big\} = 0
\end{aligned}$$

by assumption. Hence,

$$\lim_{\varepsilon \to 0} \int_0^1 \varepsilon u^{\varepsilon-1}\ Y_t(u)\ du = Y_t(0)$$

with (uniform) convergence in probability. Taking

$$Y_t(u) := \int_0^t X(s)\ \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u}\ ds$$

we obtain the assertion.

In [8] it is proved that uniform converge in probability of the Riemann–Stieltjes sums implies that in (5). Therefore our integral agrees with the Itô integral in semimartingale theory. Moreover, it also coincides with the forward integral in Malliavin calculus and with Young's approach for functions with finite p–variations (see [8]). In the next section we will show that other types of forward integrals known from the literature also fit into our model. This provides important classes of processes where the integral exists.

Extending Russo's and Vallois' notion [5] we have introduced in [8] the **quadratic variation** of a càdlàg process Z by

$$(7)\quad [Z](t) := \lim_{\varepsilon \to 0} \int_0^1 \varepsilon u^{\varepsilon - 1} \int_0^t \frac{\big(Z_{t-}(s+u) - Z_{t-}(s)\big)^2}{u}\ ds\ du + \big(Z(t) - Z(t-)\big)^2$$

whenever the limit exists (uniformly) in probability. Note that for semimartingales this agrees with the bracket used in the literature.

The covariation process of two such processes is defined similarly. (See [5] and [8] for some Itô calculus.)
Any process with generalized bracket is an element of the function space $W_2^{1/2-}$ w.p.1 (see [8]).

3. Special cases of (stochastic) forward integrals

The background from fractional calculus for our notion (5) is the following representation:

$$\frac{1}{\Gamma(\varepsilon)} \int_0^\infty u^{\varepsilon-1} \int_0^t X(s)\ \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u}\ ds\ du = \int_0^t I_{0+}^{\varepsilon}\ X(s)\ dZ(s)$$

for (random) functions X and Z possessing fractional derivatives in L_p, resp. L_q, of all orders less than α, resp. $1-\alpha$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $0 \le \alpha \le 1$ (see [7], [8]).

The integral on the right–hand side is determined by means of such derivatives:

$$\int_0^t I_{0+}^{\varepsilon}\ X(s)\ dZ(s) := (-1)^{\alpha-\varepsilon/2} \int_0^t D_{0+}^{\alpha-\varepsilon/2} X_{0+}(u)\ D_{t-}^{1-\alpha-\varepsilon/2} Z_{t-}(u)\ du + X(0+)\big(Z(t-) - Z(0)\big)\,. \tag{8}$$

(I_{0+}^{ε} stands for the fractional Riemann–Liouville integral of order ε.)
Instead of the limit $\lim\limits_{\varepsilon\to 0} \frac{1}{\Gamma(\varepsilon)} \int\limits_0^\infty u^{\varepsilon-1}\ \ldots\ du$ it is convenient to consider $\lim\limits_{\varepsilon\to 0}\ \varepsilon\ \int_0^1 u^{\varepsilon-1}\ \ldots\ du$ as above. If the marginal values α and $1-\alpha$ are included we get the function spaces $I^{\alpha}(L_p)$ and $I^{1-\alpha}(L_q)$ where the limit exists and agrees with (8) for $\varepsilon = 0$.
In particular, the stochastic integrals in equations (1) and (4) for $j = 1, \ldots, m$ may be interpreted pathwise in that sense. The stochastic integral w.r.t the process Z^0 in (1) is defined by (5), where $[Z^0]$ is given assording to (7) (continuous case).

A further advantage of the averaging procedure in the limit for small u consists in extending the following procedure:
In [2] Bedford and Kamae considered the **Cesàro averages** (in the sense of erodic theory)

$$\int_0^t X\ dZ := \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^t X(s)\ \frac{Z_{t-}(s+e^{-v}) - Z_{t-}(s)}{e^{-v}}\ ds\ dv \tag{9}$$

(for continuous X and Z) and proved existence of the (uniform) limit for certain X and self–affine functions Z. Replacing deterministic convergence by (uniform) **convergence in probability** one obtains the stochastic version for càglàd X and càdlàg Z.

Proposition. *If the stochastic integral in* (9) *is determined then it agrees with that in our definition* (5).

Proof. The right–hand side of (9) is equal to

$$\lim_{\delta\to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \frac{1}{u} \int_0^t X(s)\ \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u}\ ds\ du \tag{10}$$

so that the difference between (9) and(5) consists only in the averaging procedure as $u \searrow 0$. The kernel $\varepsilon u^{\varepsilon-1}$ in (5) arises from the last kernel $\frac{1}{|\ln\delta|}\ 1_{(\delta,1)}(u)\ \frac{1}{u}$ by means of a second averaging kernel $k_\varepsilon(v) := \varepsilon^2 v^{\varepsilon-1}|\ln v|$ on (0,1) as follows:

Denote the above inner integral by $Y_t(u)$ and use Fubini in order to get

$$\int_0^1 k_\varepsilon(v)\ \frac{1}{|\ln v|}\int_v^1 \frac{1}{u}\ Y_t(u)\ du\ dv = \int_0^1 \frac{1}{u}\int_0^u \frac{1}{|\ln v|}\ k_\varepsilon(v)\ dv\ Y_t(u)\ du$$

$$= \int_0^1 \varepsilon u^{\varepsilon-1}\ Y_t(u)\ du\,.$$

Since $\lim\limits_{v\searrow 0}\ \frac{1}{|\ln v|}\int_v^1 \frac{1}{u}\ Y_t(u)\ du = \int_0^t X\ dZ$ in the sense of Bedford and Kamae (stochastic version) and $k_\varepsilon(v)$ acts as the δ–function as $\varepsilon \to 0$ we infer similarly as in section 2:

$$\lim_{\varepsilon\to 0}\int_0^1 k_\varepsilon(v)\ \frac{1}{|\ln v|}\int_v^1 \frac{1}{u}\ Y_t\ du\ dv = \int_0^t X\ dZ$$

(uniformly) in probability. In view of the above equations this agrees with

$$\lim_{\varepsilon\to 0}\int_0^1 \varepsilon u^{\varepsilon-1}\ Y_t(u)\ du\,.$$

□

The last procedure may be applied to statistically self–similar random processes.

A further **specification** is a modification of a forward integral introduced by Klingenhöfer (personal communication):

(11) $$\int_0^t X\ dZ := \lim_{\delta\to 0}\frac{1}{\ln 2}\int_\delta^{2\delta}\frac{1}{u}\int_0^t X(s)\ \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u}\ ds\ du$$

(Klingenhöfer considered $\lim\limits_{n\to\infty} 2^{2(n+1)}\int_{2^{-(n+1)}}^{2^{-n}}\int_0^t X(s)\ \big(Z(s+u) - Z(s)\big)\ ds\ du$ in the deterministic case and proved existence for optimal Besov spaces). We here again take (uniform) convergence in probability, càglàd X and càdlàg Z.

Proposition. *If the integral in* (11) *exists then it agrees with that in the Cesàro averaging* (10)*.*

Proof. We have for $Y_t(u) := \int\limits_0^t X(s)\ \frac{Z_{t-}(s+u)-Z_{t-}(s)}{u}\ ds$

$$\frac{1}{|\ln\delta|}\int\limits_\delta^1 \frac{1}{u}\ Y_t(u)\ du = \frac{1}{|\ln\delta|}\sum_{k=1}^{n-1}\frac{1}{\ln 2}\int\limits_{2^k\delta}^{2^{k+1}\delta}\frac{1}{u}\ Y_t(u)\ du + \frac{1}{|\log\delta|}\ \frac{1}{\ln 2}\int\limits_{2^n\delta}^{1}\frac{1}{u}\ Y_t(u)\ du$$

where $n = n(\delta)$ is the integral part of $|\ln\delta|/\ln 2$. The last summand tends to 0 as $\delta \to 0$ uniformly in t. The first summand up to the factor $\frac{n}{|\ln\delta|}$ equals

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\ln 2}\int\limits_{2^{-k}2^n\delta}^{2\cdot 2^{-k}2^n\delta}\frac{1}{u}\ Y_t(u)\ du\,.$$

Denote the k-th summand of the last sum by $A_t^{n,\delta}(k)$. Using (11) we infer

$$\lim_{k\to\infty} A_t^{n,\delta}(k) = \int\limits_0^t X\ dZ$$

uniformly (in t and) in n, δ such that $n > k,\ 2^n\delta < 1$ in probability. This implies

$$\lim_{\delta\to 0}\ \frac{1}{n(\delta)}\sum_{k=1}^{n(\delta)} A_t^{n(\delta),\delta}(k) = \int\limits_0^t X\ dZ$$

(uniformly in t) in probability. Hence, the last integral agrees with the (uniform) limit in probability

$$\lim_{\delta\to 0}\ \frac{1}{|\ln\delta|}\int\limits_0^1 \frac{1}{u}\ Y_t(u)\ du\,.$$

□

Thus, we have obtained the following **implications of integrability** in the above forward integrals:

$$(6) \Rightarrow (11) \Rightarrow (9) \Rightarrow (5).$$

Using a duality result of Triebel [6] we now will provide **optimal classes of Besov spaces** for which the deterministic variant of Russo's and Vallois' integral (6) makes sense. (Recall that the norm on the Besov space $B^\alpha_{p,q}$ on $\mathbb{R}$ for the classical case $0<\alpha<1$ and $1\le p\le\infty$ may be introduced by

$$\|f\|^\alpha_{p,q} := \|f\|_{L_p} + \Big(\int_{-1}^{1} \frac{1}{|h|}\,\frac{1}{|h|^{\alpha q}}\,\Big(\|f\big((\cdot)+h\big)-f\|_{L_p}\Big)^q\,dh\Big)^{1/q}$$

if $1\le q<\infty$ and

$$\|f\|^\alpha_{p,\infty} := \|f\|_{L_p} + \sup_{|h|\le 1}\frac{1}{|h|^\alpha}\,\big\|f\big((\cdot)+h\big)-f\big\|_{L_p}\,.$$

For the quasi–normed distribution spaces $B^\alpha_{p,q}$ with general $\alpha\in\mathbb{R},\ p>0,\ q>0$ see [6].)

Theorem. *Suppose* $f\in B^\alpha_{p,q},\ g\in B^{1-\alpha}_{p',q'},\ 1\le p<\infty,\ 1\le q<\infty,\ 1/p+1/p'=1,\ 1/q+1/q'=1,\ 0\le\alpha\le 1$. *Then the limit*

$$\int f\ dg := \lim_{u\searrow 0}\int f(s)\ \frac{g(s+u)-g(s)}{u}\ ds$$

exists and we have for some constant $c>0$,

$$\Big|\int f\ dg\Big| \le c\ \|f\|^\alpha_{p,q}\ \|g\|^{1-\alpha}_{p',q'}\,.$$

Proof. Note that the Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞–functions of $\mathbb{R}$ is dense in $B^{1-\alpha}_{p',q'}$. If $g\in\mathcal{S}$ and g' denotes the derivative we obtain

$$\int f(s)\ \frac{g(s+u)-g(s)}{u}\ ds = \int f(s)\int_0^1 g'(s+ru)\ dr\ ds$$

so that for $0<\tilde u<u$,

$$\Big|\int f(s)\ \frac{g(s+u)-g(s)}{u}\ ds - \int f(s)\ \frac{g(s+\tilde u)-g(s)}{\tilde u}\ ds\Big|$$
$$\le \int_0^1\int \big|f(s)\big|\ \big|g'(s+ru)-g'(s+r\tilde u)\big|\ ds\ dr.$$

According to [6], 2.11.2, the inner integral may be estimated from above by

$$c'\ \|f\|^\alpha_{p,q}\ \|g'\big((\cdot)+ru\big)-g'\big((\cdot)+r\tilde u\big)\|^{-\alpha}_{p',q'}$$

for some constant c'. (g' is here interpreted as a distribution.)
Further, $\|\phi'\|^{-\alpha}_{p',q'}\le c''\ \|\phi\|^{1-\alpha}_{p',q'},\ \phi\in\mathcal{S}$, for some constant c'' (see, e.g., [6], 2.3.8). Hence, the above difference does not exceed

$$c\ \|f\|^\alpha_{p,q}\int_0^1 \big\|g\big((\cdot)+ru\big)-g\big((\cdot)+r\tilde u\big)\big\|^{1-\alpha}_{p',q'}\ dr$$

where $c = c'c''$. This estimation extends to arbitrary $g \in B^{1-\alpha}_{p',q'}$ via approximation by functions from $\mathcal{S}$. Similarly,

$$\left| \int f(s) \, \frac{g(s+u) - g(s)}{u} \, ds \right| \leq c \, \|f\|^{\alpha}_{p,q} \int_0^1 \left\|g\big((\cdot) + ru\big)\right\|^{1-\alpha}_{p',q'} \, dr \, .$$

Thus, it remains to show that for $1 \leq p, q \leq \infty$,

$$\lim_{\Delta \to 0} \left\|\phi\big((\cdot) + \Delta\big) - \phi\right\|^{\alpha}_{p,q} = 0$$

provided $\phi \in B^{\alpha}_{p,q}$. This can be seen as follows. Given $\varepsilon > 0$ choose $\tilde{\phi} \in \mathcal{S}$ such that $\|\phi - \tilde{\phi}\| < \varepsilon$. Then we obtain by the invariance of the norm under translations

$$\begin{aligned} \left\|\phi\big((\cdot) + \Delta\big) - \phi\right\|^{\alpha}_{p,q} &\leq \left\|\phi\big((\cdot) + \Delta\big) - \tilde{\phi}\big((\cdot) + \Delta\big)\right\|^{\alpha}_{p,q} \\ &\quad + \left\|\tilde{\phi}\big((\cdot) + \Delta\big) - \tilde{\phi}\right\|^{\alpha}_{p,q} + \left\|\tilde{\phi} - \phi\right\|^{\alpha}_{p,q} \\ &< \left\|\tilde{\phi}\big((\cdot) + \Delta\big) - \tilde{\phi}\right\|^{\alpha}_{p,q} + 2\varepsilon \, . \end{aligned}$$

Hence, it suffices to prove that the last norm tends to zero as $\Delta \to 0$. For the L_p–part of the norm this is obvious. The remaining part (up to modification for $p = \infty$ or $q = \infty$ or $\alpha = 0, 1$) is equal to

$$\begin{aligned} &\left(\int_{-1}^{1} \frac{1}{|h|^{\alpha q + 1}} \left(\left\| \int_0^{\Delta} \left(\tilde{\phi}'(x+h+r) - \tilde{\phi}'(x+r)\right) dr \right\|_{L_p} \right)^{q} dh \right)^{1/q} \\ &\leq \left\|\tilde{\phi}''\right\|_{L_\infty} \Delta \left(\int_{-1}^{1} |h|^{q(1-\alpha)-1} \, dh \right)^{1/q} \\ &= \left\|\tilde{\phi}''\right\|_{L_\infty} \operatorname{const}(\alpha, q) \, \Delta \, . \end{aligned}$$

□

Remarks. *1. The last theorem extends a partial result of* [3], *V. 3, for $q = 1$ and some p and α.*

Our conditions on the Besov spaces are best possible since we have used implicitly the duality $(B^{\alpha}_{p,q})' = B^{-\alpha}_{p',q'}$. By the norm estimators the forward integral in this case agrees with the continuous extension of the Lebesgue–Stieltjes integral to these function spaces.

2. The ideas of proof remain valid for the stochastic forward integral (5) *if the Besov spaces are replaced by some stochastic variants. This will be shown elsewhere. For the method of fractional derivatives introduced in* [7], [8] *and fractional Brownian motion as integrator such a combination of analysis and stochastics is used in* [1], *section 5.*

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