SEMIGROUPS, POTENTIAL SPACES AND APPLICATIONS TO (S)PDE

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Abstract. The paper studies perturbed semilinear parabolic partial (pseudo-) differential equations on σ-finite measure spaces under low smoothness assumptions. We obtain results on existence, uniqueness and regularity. The hypotheses are formulated in terms of the semigroup, regularity is measured by means of abstract potential spaces. Being a priori analytic, our results allow to investigate related stochastic partial differential equations in the almost sure pathwise sense. For example we can study (fractional) semilinear heat equations driven by fractional Brownian noises on metric measure spaces.

1. Introduction

The aim of this article is to study parabolic partial (pseudo-) differential equations on fairly general spaces. We are interested in a theory that needs few assumptions but makes sense both for PDE and SPDE. For example Cauchy problems associated to perturbed semilinear equations like

$$\begin{cases} \frac{\partial u}{\partial t} = -A^\theta u + F(u) + G(u)\cdot \dot{z} \ \ , \ t\in(0,t_0) \\ u(0)=f \end{cases} \tag{1}$$

are considered on general *σ-finite measure spaces* $(X,\mathcal{X},\mu)$, including *fractal metric measure spaces* such as Sierpinski gaskets or carpets. Here $t_0>0$ and $0<\theta\le 1$ may be chosen arbitrarily. $-A$ is the generator of a symmetric Markovian semigroup on $L_2(\mu)$, for $0<\theta\le 1$ the operator A^θ is a fractional power of A, and F, G are sufficiently regular functions, seen as composition operators. $\dot{z}$ denotes a non-differentiable space-time perturbation, so far deterministic.
We interpret the formal problem (1) in the *mild (evolution) sense* (2) below and, under suitable hypotheses, show *existence, uniqueness and regularity of function solutions*. Then similar results also hold for related stochastic equations in the pathwise sense with $\dot{z}$ replaced by a typical realization of a random noise.

Deterministic elliptic or parabolic (pseudo-) differential equations on (metric) measure spaces had been investigated in [16], see also [2], [27] and [36].

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Many results can already be obtained in abstract functional analytic formulation, cf. e.g. [32]. Of course also SPDE on general measure spaces can be studied that way, see [14], [33] for abstract problems with Brownian and [31], [39] for such with fractional Brownian noises. In our pathwise approach [22] the definition of the integral had been slightly different (based on fractional calculus and independent of a series expansion), what led to efficient results and allowed a broad variety of noises. A related construction had been used in [18], giving basically the same results. We refer to [22] for a detailed discussion.
In the classical random field formulation, [13], [42], stochastic partial (pseudo-) differential equations on fractal spaces had occured as explicit examples in the paper [29]. There the noise had been taken to be additive and white in time and space. It had been shown that function solutions to such equations exist if and only if a symmetrized version of the Markov process associated with the generator has local times. For instance, if the generator is chosen to be the standard (Dirichlet) Laplacian on a fractal metric measure space (X, d, μ), [2], [27], this is the case provided the spectral dimension d_S is strictly less than two, [2], Theorem 3.32.

Here we are interested in function solutions to semilinear equations with multiplicative noise fractional in time and space. As general fractional noises lack semimartingale properties, Itô type integrals as in [14], [33], respectively [13], [42] do not apply. We adapt the pathwise approach of [21], [22] to the present situation. In particular, (1) is interpreted as deterministic evolution problem in abstract *potential spaces*. Their fractional order is used to encode the spatial regularity of the driving and the prospective solution, the temporal regularity is measured by Hölder type norms. The abstract duals of potential spaces are seen as substitutes for spaces containing distributions with negative smoothness. For classical potential spaces on Euclidean domains with applications to partial differential equations we refer to [17], [37] and [40]. In [24], [25] interesting non-classical potential spaces had been obtained from strongly continuous symmetric Markovian semigroups on metric measure spaces.
We start from the strongly continuous symmetric Markovian semigroup generated by a Dirichlet operator $-A$, see [9]. The interpretation of related low order potential spaces as *Dirichlet spaces* simplifies the definition and evaluation of composition operators and pointwise products, problems that arise naturally in a pathwise study of stochastic partial differential equations with multiplicative noise. Under the additional assumption that the semigroup is ultracontractive, basic results on composition and multiplication already follow from the Markov property and allow to prove Theorem 1.1 and Corollary 1.1. Arguments of this type have been around for some time, see the examples in [31] and [18]. They can be worked out for an arbitrary measure space $(X, \mathcal{X}, \mu)$ and do neither need the presence of a topology nor specific information about the structure of the associated Dirichlet forms.

A function u on $[0,t_0]\times X$ is called a *mild solution* to (1) if seen as vector-valued function $u(t) := u(t,\cdot)$, it satisfies

$$(2)\quad u(t) = T^{(\theta)}(t)f + \int_0^t T^{(\theta)}(t-s)F(u(s))ds + \int_0^t T^{(\theta)}(t-s)G(u(s))dz(s),\quad t\in(0,t_0)\ ,$$

where $T^{(\theta)} = (T^{(\theta)}(t))_{t\geq 0}$ is the subordinated semigroup associated with the fractional power $-A^\theta$ and the last summand denotes a deterministic integral operator as in [22], see Section 5 below. If for any fixed $t\in[0,t_0]$, $u(t)$ determines a locally integrable function on $(X,\mathcal{X},\mu)$, we call it a *function solution*. A temporal differentiation of z is realized by means of fractional calculus, a spatial differentiation is hidden in the fact that for fixed time s, $z(s)$ is an element of the dual of an appropriate potential space. In (1) we write $\dot{z}$ for the formal time derivative of z. Applied to stochastic problems in the pathwise sense this construction yields an *integral of Stratonovich type*.
We follow the schedule of [22] and establish the contractivity of the integral operator in Sobolev type spaces of functions on $[0,t_0]$ that take their values in suitable potential spaces on X. Existence and uniqueness of mild solutions are then implied by *Banach's contraction principle*. We obtain the following results.

Case I: Nonlinear multiplicative noise terms, general measure spaces.

For the definitions of the spaces $H^\sigma(\mu)$, $H^\sigma_\infty(\mu)$ and $H^{-\sigma}(\mu)$ we refer to Sections 2, in particular to (14) and (15). The Hölder and Sobolev spaces $C^\eta([0,t_0],H^{-\sigma}(\mu))$ and $W^\eta([0,t_0],H^\sigma_\infty(\mu))$ of functions on $[0,t_0]$ with values in these spaces are formally introduced in Section 5.

Theorem 1.1. *Assume $(X,\mathcal{X},\mu)$ is a σ-finite measure space, $t_0>0$ and $0<\theta\leq 1$. Let $-A$ be the generator of a strongly continuous symmetric Markovian semigroup $(T(t))_{t\geq 0}$ on $L_2(\mu)$ which is ultracontractive with spectral dimension $d_S>0$.*
Suppose $0<\alpha,\beta,\gamma,\delta,\varepsilon<1$ and $z\in C^{1-\alpha}([0,t_0],H^{-\beta\theta}(\mu))$. Let $F\in C^1(\mathbb{R}^n)$, $F(0)=0$, have a bounded Lipschitz derivative F' and $G\in C^2(\mathbb{R})$, $G(0)=0$, have a bounded Lipschitz second derivative G''. Assume $f\in H^{2\gamma+\delta\theta+\varepsilon}(\mu)$. If $\alpha<\gamma<1-\alpha$, $\delta\geq\beta$ and

$$(3)\qquad 2\gamma + \left(\delta\vee\frac{d_S}{2\theta}\right) < 2-2\alpha-\left(\beta\vee\frac{d_S}{2\theta}\right)\ ,$$

then problem (1) has a unique mild solution (2) in $W^\gamma([0,t_0],H^{\delta\theta}_\infty(\mu))$.

The spectral dimension $d_S>0$ can be read off from decay bounds for the semigroup, see Section 3 or for instance [41].
We remark that Theorem 1.1 remains true if z is taken from the slightly larger space $C^{1-\alpha}([0,t_0],(H^{\beta\theta}_\infty(\mu))^*)$.

Corollary 1.1. *Assume the hypotheses of Theorem 1.1 hold and A has pure point spectrum. Let $\{\varphi_j\}_{j=1}^{\infty}$ be the complete orthonormal system of eigenfunctions of A corresponding to the eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \dots$ and let*

$$z(t) = \sum_{j=1}^{\infty} q_j \beta_j^H(t) \varphi_j \ ,$$

where $\left\{\beta_j^H\right\}_{j=1}^{\infty}$ is a family of mutually independent real valued fractional Brownian motions with Hurst index $1-\alpha < H < 1$ and $\{q_j\}_{j=1}^{\infty}$ is a sequence of reals such that $\sum_{j=1}^{\infty} q_j^2 \lambda^{-\beta\theta} < \infty$.
Then almost surely $z \in C^{1-\alpha}([0,t_0], H^{-\beta\theta}(\mu))$ and the stochastic problem (1) has a unique mild solution (2) in $W^{\gamma}([0,t_0], H_{\infty}^{\delta\theta}(\mu))$.

In the case $\mu(X) < \infty$ it follows from our hypotheses that A has pure point spectrum, see Section 3.

Theorem 1.1 is very general but not efficient, compare it for instance to Theorem 4.2 in [22]. We will also investigate a second set of hypotheses where (X, μ, d) is a metric measure space and the semigroup is additionally assumed to possess transition densities that satisfy typical estimates. In such cases the results on pointwise multiplication can be refined significantly and lead to Theorems 1.2 and 1.3 below. These statements allow for a wider range of parameters, in particular the spectral dimension d_S may be larger. We can then fully recover and extend the results previously shown in [22] for Euclidean domains.

Assumption (MMS):
(X,d) is a locally compact separable metric space, $\mathcal{X} = \mathcal{B}(X)$ the Borel-σ-field on X and μ a Radon measure on (X,d).

Let $(P_t^{(\theta)}(x,dy))_{t\ge 0}$ be the symmetric transition kernel associated with $T^{(\theta)} = (T^{(\theta)}(t))_{t\ge 0}$,

$$T^{(\theta)}(t)u(x) = \int_X u(y) P_t^{(\theta)}(x,dy) \ . \tag{4}$$

θ is suppressed from notation if it equals 1.

Assumption (HKE(β)):
$(P_t(x,dy))_{t\ge 0}$ admits transitions densities, $P_t(x,dy) = p(t,x,y)\mu(dy)$, which satisfy the following heat kernel estimates:

$$t^{-d_f/w}\Phi_1(t^{-1/w}d(x,y)) \le p(t,x,y) \le t^{-d_f/w}\Phi_2(t^{-1/w}d(x,y))$$

for any $(x,y) \in X \times X$ and $t \in (0,R_0)$, with bounded decreasing functions Φ_i on $[0,\infty)$. Here $R_0 > 0$ is a fixed number. d_f is the Hausdorff-Dimension of (X,d) and $w \ge 2$ satisfies $d_S = \frac{2d_f}{w}$. For a given number $\beta > 0$ we further assume the validity of the integral condition

$$\int_0^\infty s^{d_f+\beta/2-1}\Phi_2(s)ds < \infty \ .$$

In these notations we get the following:

Case II: Nonlinear multiplicative noise terms, metric measure spaces.

The spaces $H_q^{-\sigma}(\mu)$ are also defined in Section 2.

Theorem 1.2. *Let t_0, θ, $-A$, $(T(t))_{t\geq 0}$, F and G be as in Theorem 1.1. Suppose $0 < \alpha, \beta, \gamma, \delta, \varepsilon < 1$ and (MMS) and (HKE(β)) hold. Assume $\alpha < \gamma < 1-\alpha$ and $0 < \beta < \delta < \frac{d_S}{2\theta}$. If $f \in H^{2\gamma+\delta\theta+\varepsilon}(\mu)$ and $z \in C^{1-\alpha}([0,t_0], H_q^{-\beta\theta}(\mu))$ for $q = \frac{d_S}{\delta\theta}$ and*

$$2\gamma + \frac{d_S}{2\theta} < 2 - 2\alpha - \beta \ , \tag{5}$$

then problem (1) has a unique mild solution (2) in $W^\gamma([0,t_0], H_\infty^{\delta\theta}(\mu))$.

So far the theory is low dimensional, in Theorem 1.2 we must necessarily have $d_S < 4\theta$. This condition stems from the singularity of the semigroup at zero and subordination leads to a stronger restriction. For symmetric diffusion semigroups on $\mathbb{R}^n$ we have $d_S = n$, hence need $n \leq 3$.
However, in the special case that F and G are linear, arguments involving the $L_\infty(\mu)$-boundedness of the solution are no longer needed and we can dispose the dimension restriction:

Case III: Linear multiplicative noise terms, metric measure spaces.

Theorem 1.3. *Let t_0, θ, $-A$, $(T(t))_{t\geq 0}$ be as in Theorem 1.1 and let F, G be linear. Suppose $0 < \alpha, \beta, \gamma, \delta, \varepsilon < 1$ and (MMS) and (HKE(β)) hold. Assume $\alpha < \gamma < 1-\alpha$ and $0 < \beta < \delta < \frac{d_S}{2\theta}$. If $f \in H^{2\gamma+\delta\theta+\varepsilon}(\mu)$ and $z \in C^{1-\alpha}([0,t_0], H_q^{-\beta\theta}(\mu))$ for $q = \frac{d_S}{\delta\theta}$ and*

$$2\gamma + \delta < 2 - 2\alpha - \beta \ , \tag{6}$$

then problem (1) has a unique mild solution (2) in $W^\gamma([0,t_0], H^{\delta\theta}(\mu))$.

In the next section, we provide some background on semigroups and potential spaces on general σ-finite measure spaces. In Section 3 we use Dirichlet forms to obtain basic pointwise multiplication and composition properties. Under the assumptions (MMS) and (HKE(β)) refined multiplication properties are then proved in Section 4.
Section 6 finally studies parabolic (fractional) Cauchy problems. In Section 7 we highlight a few examples.

2. Semigroups and potential spaces

Let $(X, \mathcal{X}, \mu)$ be a σ-finite measure space. $L_\infty(\mu)$ denotes the space of all real valued essentially bounded measurable functions on X, equipped with its usual (supremum) norm $\|\cdot\|_{L_\infty(\mu)}$, and $L_p(\mu)$, $1 \le p < \infty$, the space of all real valued measurable and integrable functions with its usual norm $\|\cdot\|_{L_p(\mu)}$. In the case $p = 2$ we also use the shortcuts $\|\cdot\|_0$ and $(\cdot,\cdot)_0$ for the norm and the scalar product.

Assume that $T = (T(t))_{t\ge 0}$ is a strongly continuous semigroup on $L_2(\mu)$, i.e. $T(0) = I$, I denoting the identity operator, $T(t+s) = T(t)\circ T(s)$, $t, s \ge 0$ and $\lim_{t\to 0}\|T(t)u - u\|_0 = 0$ for any $u \in L_2(\mu)$. Let $-A$ denote the infinitesimal generator of $(T(t))_{t\ge 0}$,

$$(7) \qquad -Au = \lim_{t\to 0}\frac{1}{t}(T(t)u - u) \ \text{ strongly in } L_2(\mu)$$

for members u of $\mathcal{D}(A)$, the space of all $u \in L_2(\mu)$ such that this limit exists. $\mathcal{D}(A)$ is dense in $L_2(\mu)$, cf. [44]. If we further suppose that $T = (T(t))_{t\ge 0}$ is symmetric, $(T(t)u, v)_0 = (u, T(t)v)_0$ for any $t > 0$ and $u, v \in L_2(\mu)$, then it follows that both A and $T(t)$ are non-negative definite self-adjoint operators on $L_2(\mu)$ with spectral representations

$$(8) \qquad A = \int_0^\infty \lambda dE_\lambda \ \text{ respectively } \ T(t) = \int_0^\infty e^{-\lambda t} dE_\lambda\ .$$

For any $\alpha \ge 0$ the fractional power A^α of A is given by

$$(9) \qquad A^\alpha = \int_0^\infty \lambda^\alpha dE_\lambda\ ,$$

equally self-adjoint and non-negative. If $0 < \alpha < 1$, we have a particularly simple description of A^α in terms of the semigroup along with a characterization of its domain $\mathcal{D}(A^\alpha)$: $u \in L_2(\mu)$ is in $\mathcal{D}(A^\alpha)$ if and only if

$$(10) \qquad A^\alpha u = \lim_{\varepsilon\to 0}\frac{1}{\Gamma(-\alpha)}\int_\varepsilon^\infty t^{-\alpha-1}(T(t) - I)u dt$$

converges in $L_2(\mu)$, see e.g. [7]. (We remark that in [22] the sign had been incorrect.) This may be interpreted as a right-sided Weyl-Marchaud derivative D_-^α of $t \mapsto T(t)u$ at $t = 0$, more precisely, $D_-^\alpha(T(\cdot)u))(t) = (-1)^\alpha A^\alpha T(t)u$. See [22], [34] and Section 5 below.

Now let us temporarily assume that zero is not an eigenvalue of A. Then the negative fractional powers

$$A^{-\alpha} = \int_0^\infty \lambda^{-\alpha} dE_\lambda\ ,$$

$\alpha > 0$ can be expressed by

$$(11) \qquad A^{-\alpha}u = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}T(t)u dt\ ,$$

what may be read as a right-sided Riemann-Liouville integral I_-^α of order $\alpha > 0$ of the function $t \mapsto T(t)u$, i.e. $I_-^\alpha(T(\cdot)u)(t) = (-1)^{-\alpha}A^{-\alpha}T(t)u$. Thus, for semigroups the language of traditional fractional calculus just leads to special cases of the usual functional calculus, cf. [34], [35], [44].

If the symmetric strongly continuous semigroup $T = (T(t))_{t\ge 0}$ is contractive,

$$\|T(t)u\|_0 \le \|u\|_0\ ,\quad t > 0,\ u \in L_2(\mu)\ ,$$

it already follows that it is analytic, cf. [15] or [35], Chapter III. In this case it enjoys a number of useful properties: For any $u \in L_2(\mu)$, $\alpha \geq 0$ and $t > 0$ we have $T(t)u \in \mathcal{D}(A^\alpha)$. The operators $T(t)$ and A^α commute on $\mathcal{D}(A^\alpha)$. Given $\omega > 0$, the bound

$$\|(\omega I + A)^\alpha T(t)\| \leq c_\alpha e^{\omega t} t^{-\alpha} \tag{12}$$

holds for $t > 0$ (in the operator norm on $L_2(\mu)$) and the continuity estimate

$$\|T(t)u - u\|_0 \leq c_\alpha t^\alpha \|(\omega I + A)^\alpha u\|_0 + (1 - e^{-\omega t}) \|f\|_0 \tag{13}$$

is valid for $0 \leq \alpha < 1$, $u \in \mathcal{D}(A^\alpha)$ and $t > 0$. Recall that $(1 - e^{-\omega t}) \leq \omega t$. If zero is not an eigenvalue, (12) and (13) also hold for $\omega = 0$, see [32].
In the analytic case also the subordinated semigroup $T^{(\alpha)} = (T^{(\alpha)}(t))_{t \geq 0}$,

$$T^{(\alpha)}(t) = \int_0^\infty e^{-\lambda^\alpha t} dE_\lambda \ ,$$

generated by $-A^\alpha$, $0 < \alpha < 1$ is analytic, [35], [44]. To save notation, we consider $0 < \alpha \leq 1$ and write $T^{(1)}(t) = T(t)$, $t \geq 0$.
It is further known that in the symmetric and contractive case $T = (T(t))_{t \geq 0}$ uniquely determines analytic semigroups on the spaces $L_p(\mu)$, $1 \leq p < \infty$, see [15], Theorem 1.4.1. or [35], Chapter III. We use the same notation $T = (T(t))_{t \geq 0}$ for these semigroups but denote their $L_p(\mu)$-generators (7) by $-A_p$, such that $A_2 = A$. In these cases we may use (10) and (11) to define their fractional powers, see [44].
Given $\alpha_1, \alpha_2 \geq 0$, we have $A^{\alpha_1 + \alpha_2} = A^{\alpha_1} A^{\alpha_2}$, $A^{\alpha_1} A^{-\alpha_1} = I$ and $A^{\alpha_1} : \mathcal{D}(A^{\alpha_1 + \alpha_2}) \to \mathcal{D}(A^{\alpha_2})$ is an isomorphism between these domains endowed with the graph norm. For $\sigma \geq 0$ we may regard the negative power

$$I^\sigma(\mu) := A^{-\sigma/2}$$

as a *generalized Riesz potential operator* on $L_2(\mu)$. If zero is an eigenvalue of A, we may instead consider $A + I$ and define a *generalized Bessel potential operator* by

$$J^\sigma(\mu) := (A + I)^{-\sigma/2} \ .$$

The corresponding *potential spaces* are given by the Hilbert spaces

$$\dot{H}^\sigma(\mu) := I^\sigma(\mu)(L_2(\mu)) \quad \text{and} \quad H^\sigma(\mu) := J^\sigma(\mu)(L_2(\mu)), \tag{14}$$

$\sigma \geq 0$, equipped with the norms

$$\|u\|_{\dot{\sigma}} := \left\|A^{\sigma/2} u\right\|_0 \quad \text{and} \quad \|u\|_\sigma := \|u\|_0 + \left\|A^{\sigma/2} u\right\|_0 \ ,$$

respectively. If zero is in the resolvent set, the spaces coincide. Note that $\mathcal{D}((I + A)^\alpha) = \mathcal{D}(A^\alpha) = H^{2\alpha}(\mu)$, $\alpha \geq 0$, and by their definition as negative fractional powers the generalized potential operators act as isomorphisms.
Above we had agreed to denote the $L_p(\mu)$-generator of $T = (T(t))_{t \geq 0}$ by $-A_p$, $1 < p < \infty$. Similarly as for $A_2 = A$ we may consider fractional powers (10), (11) of A_p and define related potential spaces by

$$\dot{H}^\sigma_p(\mu) := I^\sigma(\mu)(L_p(\mu)) \quad \text{and} \quad H^\sigma_p(\mu) := J^\sigma(\mu)(L_p(\mu)),$$

$\sigma \geq 0$, where $J^{\sigma}(\mu) = (A+I)^{-\sigma/2}$. These spaces are normed by

$$\|u\|_{p,\dot{\sigma}} := \left\|A^{\sigma/2}u\right\|_{L_p(\mu)} \quad \text{and} \quad \|u\|_{p,\sigma} := \|u\|_{L_p(\mu)} + \left\|A^{\sigma/2}u\right\|_{L_p(\mu)} .$$

Obviously $H_2^{\sigma}(\mu) = H^{\sigma}(\mu)$. Note that just as in the special case $p = 2$, potential operators $J^{\sigma}(\mu)$, $\sigma \geq 0$ define isomorphic mappings from $H_p^{\alpha}(\mu)$ onto $H_p^{\alpha+\sigma}(\mu)$, $\alpha \geq 0$. Similarly for $I^{\sigma}(\mu)$, provided zero is in the resolvent set. We also consider the subspaces

$$H_{\infty}^{\sigma}(\mu) := H^{\sigma}(\mu) \cap L_{\infty}(\mu),$$

normed by $\|\cdot\|_{\sigma,\infty} := \|\cdot\|_{\sigma} + \|\cdot\|_{L_{\infty}(\mu)}$. Clearly this is an abuse of notation, but as we have defined the spaces $H_p^{\sigma}(\mu)$ for $1 < p < \infty$ only, no confusion will occur. Denote by

$$H_{p'}^{-\sigma}(\mu) := ((H_p^{\sigma}(\mu))^* \tag{15}$$

the *duals of the spaces* $H_p^{\sigma}(\mu)$, $1 < p < \infty$, $\sigma \geq 0$, $\frac{1}{p} + \frac{1}{p'} = 1$, equipped with the usual (operator) norm $\|\cdot\|_{H_{p'}^{-\sigma}(\mu)}$. If $p = 2$, p is suppressed and we write $\|\cdot\|_{-\sigma}$ for the norm. Note that in general $H^{-\sigma}(\mu)$ is a subspace of $(H_{\infty}^{\sigma}(\mu))^*$. The above notation is only an algebraically convenient shortcut and not to be confused with its classical meaning in the theory of Schwartz distributions. However, it is in agreement with this theory if $X = \mathbb{R}^n$, $X = D$ or $X = \overline{D}$ for a smooth bounded domain $D \subset \mathbb{R}^n$. In this case the fractional Sobolev spaces relevant for initial or boundary value problems are well studied and the duals are known explicitely, cf. [37], [40]. In general there is no explicit description for $H_{p'}^{-\sigma}(\mu)$.

A nice situation arises if A has a pure point spectrum, cf. [44], and there exists a complete orthonormal system $\{\varphi_j\}_{j=1}^{\infty}$ in $L_2(\mu)$ consisting of eigenvectors φ_j of A with corresponding eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq \lambda_{n+1} \leq ...$ of finite multiplicity and $\lim_{j\to\infty} \lambda_j = +\infty$. Moreover, as the spectral representations (8), (9) become discrete, we have

$$A^{\alpha}u = \sum_{j=1}^{\infty} \lambda_j^{\alpha}(u, \varphi_j)_0 \varphi_j \tag{16}$$

for any

$$u \in H^{2\alpha}(\mu) = \left\{ u \in L_2(\mu) : \sum_{j=1}^{\infty} \lambda_j^{2\alpha} |(u, \varphi_j)_0|^2 < \infty \right\}$$

with $\alpha \geq 0$ fixed. As before we see that if $\lambda_1 > 0$, (16) makes sense also for $-\alpha$ in place of $\alpha > 0$, $A^{-\alpha}$ being bounded on $L_2(\mu)$, recall (11).

Having in mind the application to noise expansions and SPDE we note the following:

Lemma 2.1. *Assume $\sigma \geq 0$ and $\lambda_1 > 0$. A formal series expansion $v = \sum_{j=1}^{\infty} q_j \varphi_j$ with real coefficients q_j converges in $H^{-\sigma}(\mu)$ if*

$$\|v\|_{-\sigma} = \sum_{j=1}^{\infty} q_j^2 \lambda_j^{-\sigma} < \infty.$$

Proof. Let $u = \sum_{j=1}^{\infty}(u,\varphi_j)_0\varphi_j \in H^\sigma(\mu) \subset L_2(\mu)$ with $\|u\|_\sigma^2 = \sum_{j=1}^{\infty}\lambda_j^\sigma|(u,\varphi_j)_0|^2 = 1$, note that $\left\{\lambda_j^{-\sigma/2}\varphi_j\right\}_{j=1}^{\infty}$ is a complete orthonormal system in $H^\sigma(\mu)$. For the dual pairing (u,v) of u with v we then observe

$$|(u,v)| = |\sum_{j=1}^{\infty} q_j(u,\varphi_j)_0| \le \sum_{j=1}^{\infty}|q_j|\lambda^{-\sigma/2}\lambda^{\sigma/2}|(u,\varphi_j)_0| \le \left(\sum_{j=1}^{\infty} q_j^2\lambda^{-\sigma}\right)^{1/2} .$$

□

3. Basic results on multiplication and composition

A study of semilinear equations requires a definition of pointwise multiplication and knowledge about the behaviour of composition operators. To settle these questions we use the language of Dirichlet forms.
The symmetric strongly continuous $L_2(\mu)$-semigroup $T = (T(t))_{t\ge0}$ enjoys the *Markov property* if any of its operators is Markovian, that is if for any $t \ge 0$ and any $u \in L_2(\mu)$ with $0 \le u \le 1$ μ-a.e. we have

$$0 \le T(t)u \le 1 \quad \mu\text{-a.e.}$$

If so, it is automatically contractive, [9], Corollary 2.2.4. In the Markovian case we can define

$$\mathcal{E}(u) := \mathcal{E}(u,u) := \sup_{t>0}\frac{1}{t}(u - T(t)u, u)_0 \tag{17}$$

for $u \in L_2(\mu)$ and denote by $\mathcal{D}(\mathcal{E})$ the domain of $\mathcal{E}$ as the space of all $u \in L_2(\mu)$ such that (17) is finite. By polarization $\mathcal{E}$ defines a non-negative definite symmetric bilinear form on $L_2(\mu)$. According to our hypotheses it is a *Dirichlet form*, in particular, $\mathcal{D}(\mathcal{E})$ is dense in $L_2(\mu)$ and becomes a Hilbert space normed by $(\mathcal{E}(u) + \omega\,\|u\|_0^2)^{1/2}$ for any $\omega > 0$. In our case we obviously have $\mathcal{D}(\mathcal{E}) = H^1(\mu) = \mathcal{D}(A^{1/2})$ and the described norm is equivalent to the norm $\|\cdot\|_1$. Finally, the Markov property of the semigroup is equivalent to a corresponding Markov property for the Dirichlet form: For any function $F : \mathbb{R} \to \mathbb{R}$ with $F(0) = 0$ and $|F(x) - F(y)| \le |x - y|$ and for any $u \in \mathcal{D}(\mathcal{E})$,

$$\mathcal{E}(F(u)) \le \mathcal{E}(u) .$$

We refer to [9] and [17] for details. Let us remark that for $0 < \alpha \le 1$ also $T^{(\alpha)} = (T^{(\alpha)}(t))_{t\ge0}$ is Markovian whenever $T = (T(t))_{t\ge0}$ is. We can define a form $\mathcal{E}^{(\alpha)}$ as in (17) with $T(t)$ replaced by $T^{(\alpha)}(t)$. Then also $\mathcal{E}^{(\alpha)}$ is a Dirichlet form, $\mathcal{D}(\mathcal{E}^{(\alpha)})$ coincides with $H^\alpha(\mu)$ and $(\mathcal{E}^{(\alpha)}(u) + \|u\|_0^2)^{1/2}$ provides an equivalent norm in that space. Of course $\mathcal{E}^{(1)} = \mathcal{E}$.
From the Markov property of $(T^{(\sigma)}(t))_{t\ge0}$ respectively $\mathcal{E}^{(\sigma)}$, $0 < \sigma \le 1$ it follows that $H^\sigma_\infty(\mu)$ is a *multiplication algebra*, more precisely,

$$(\mathcal{E}^{(\sigma)}(uv))^{1/2} \le \|u\|_{L_\infty(\mu)}(\mathcal{E}^{(\sigma)}(v))^{1/2} + \|v\|_{L_\infty(\mu)}(\mathcal{E}^{(\sigma)}(u))^{1/2}$$

for $u, v \in H^\sigma_\infty(\mu)$, $0 < \sigma \le 1$ and therefore

$$\|uv\|_{\sigma,\infty} \le c\,\|u\|_{\sigma,\infty}\,\|v\|_{\sigma,\infty} \ . \tag{18}$$

We refer to [9], Proposition 2.3.2 and Corollary 3.3.2.
Combining duality and multiplication we can define *products of functions and dual elements* as follows. For $u \in H^\sigma_\infty(\mu)$ and $z \in (H^\sigma_\infty(\mu))^*$, $0 < \sigma \leq 1$ define the product uz by

$$(v, uz) := (vu, z) \quad , \quad v \in H^\sigma_\infty(\mu) \ , \tag{19}$$

recall that $(\cdot, \cdot)$ denotes the dual pairing. Then uz is well defined as an element in $(H^\sigma_\infty(\mu))^*$ and by (18),

$$\|uz\|_{(H^\sigma_\infty(\mu))^*} \leq \|z\|_{(H^\sigma_\infty(\mu))^*} \|u\|_{\sigma,\infty} \ , \tag{20}$$

since for any $v \in H^\sigma_\infty(\mu)$,

$$|(v, uz)| = |(vu, z)| \leq \|z\|_{(H^\sigma_\infty(\mu))^*} \|uv\|_{\sigma,\infty} \ .$$

Let us return to the semigroup. If there are constants $c > 0$, $0 < \omega < 1$ and $d_S > 0$ such that for any $t > 0$,

$$\|T(t)\|_{L_2(\mu)\to L_\infty(\mu)} \leq ct^{-d_s/4}e^{\omega t} \ , \tag{21}$$

we say that $T = (T(t))_{t\geq 0}$ is *(locally) ultracontractive with spectral dimension* d_S, cf. [2], [10], a property equivalent to the validity of various functional inequalities, see for instance [10], [12], [41]. One is Nash's inequality,

$$\|u\|_0^{2+4/d_S} \leq c\,(\mathcal{E}(u) + \omega\,\|u\|_0^2)\,\|u\|_{L_1(\mu)}^{4/d_s}\ ,\ u \in H^1(\mu)\,. \tag{22}$$

For $d_S > 4$ (22) is also equivalent to the Sobolev embedding

$$\|u\|^2_{L_{2d_S/(d_S-4)}} \leq c\,(\mathcal{E}(u) + \omega\,\|u\|_0^2),\ u \in H^1(\mu)\,. \tag{23}$$

The equivalences of (21) and (22) resp. (23) hold in general σ-finite measure spaces, cf. [12]. Note that Nash's inequality (22) and the bound (21) remain valid for the subordinated semigroup $T^{(\alpha)} = (T^{(\alpha)})_{t\geq 0}$ and Dirichlet form $\mathcal{E}^{(\alpha)}$, $0 < \alpha \leq 1$, then with $\frac{d_S}{\alpha}$ in place of d_S. See [6].
There is another pleasant fact, for a proof see [15], Lemma 2.1.2 and Theorem 2.1.4 or [43], Theorems 3.2 and 4.5:

Lemma 3.1. *Assume $\mu(X) < \infty$ and that $(T(t))_{t\geq 0}$ is strongly continuous, symmetric and Markovian. If in addition it is ultracontractive with spectral dimension $d_S > 0$, then A has pure point spectrum.*

If (21) holds for $(T(t))_{t\geq 0}$, we can use its symmetry to define $T(t)z$ even for elements $z \in (H^\sigma_\infty(\mu))^*$, $0 < \sigma \leq 1$. For $t > 0$ let $T(t)z$ be the element of $L_2(\mu)$ which is determined by the duality relation

$$(v, T(t)z) := (T(t)v, z) \quad , \quad v \in L_2(\mu) \ .$$

Note that by analyticity $T(t)v \in H^\alpha_\infty(\mu)$ for any $\alpha \geq 0$. From

$$|(v, T(t)z)| = |(T(t)v, z)| \leq \|T(t)v\|_{\sigma,\infty} \|z\|_{(H^\sigma_\infty(\mu))^*}$$

together with (12) and (21) we deduce the estimate

$$\|T(t)z\|_0 \leq c(t^{-\sigma/2} + t^{-d_S/4})\,\|z\|_{(H^\sigma_\infty(\mu))^*} \tag{24}$$

for any $t \in (0, t_0)$ and $z \in (H^\sigma_\infty(\mu))^*$.

Remark 3.1.

(i) Recall that $H^{-\sigma}(\mu)$ is continuously embedded and in general strictly contained in $(H^\sigma_\infty(\mu))^*$.

(ii) If $\sigma > d_S/2$, one can alternatively use the mapping properties of $I^\sigma(\mu)$ (or $J^\sigma(\mu)$) in conjunction with duality to explain the action of $T(t)$ on $(H^\sigma_\infty(\mu))^*$. This requires the boundedness of $I^\sigma(\mu)$: $H^\alpha_\infty(\mu) \to H^{\alpha+\sigma}_\infty(\mu)$ (resp. $J^\sigma(\mu)$), $\alpha \geq 0$, which in this case is guaranteed by (11) .

We provide key estimates for composition operators. Let $F : \mathbb{R} \to \mathbb{R}$ be a C^1-function with bounded Lipschitz derivative F' and such that $F(0) = 0$. Let $G : \mathbb{R} \to \mathbb{R}$ be a C^2 function with bounded and Lipschitz second derivative G'' and $G(0) = 0$. The associated composition operators $u \mapsto F(u)$ and $u \mapsto G(u)$ are also denoted by F and G.

Proposition 3.1. *Let $0 < \sigma \leq 1$. F and G are bounded from $H^\sigma_\infty(\mu)$ into itself and the following estimates hold:*

(i) *Given $u, v \in H^\sigma_\infty(\mu)$ we have*

$$\|F(u) - F(v)\|_{\sigma,\infty} \leq c\, \|u - v\|_{\sigma,\infty} (\|v\|_{\sigma,\infty} + 1).$$

Similarly for G in place of F.

(ii) *Given $u_i, v_i \in H^\sigma_\infty(\mu)$ with $\|u_i\|_{\sigma,\infty} \leq 1$, $\|v_i\|_{\sigma,\infty} \leq 1$, $i = 1, 2$,*

$$\|G(u_1) - G(v_1) - G(u_2) + G(v_2)\|_{\sigma,\infty} \leq c\, (\|u_1 - v_1 - u_2 + v_2\|_{\sigma,\infty} + \|u_2 - v_2\|_{\sigma,\infty}).$$

The Lipschitz conditions on F' and G'' can be weakened if one is willing to work with the respective moduli of continuity, see for instance [46]. Though it mainly consists of standard estimates, see Proposition 3.3.1 in [9] and Proposition 2.1 in [46], we give a proof of Proposition 3.1. It finishes the present section.

$L_\infty(\mathbb{R})$ denotes the space of Lebesgue measurable and essentially bounded functions on the real line.

Proof. Step 1. We begin with recording some elementary estimates on F, seen as real-valued function on the line. The hypotheses together with the mean value theorem yield

$$\begin{aligned}
&|F(w) - F(x) - F(y) + F(z)| \\
&= |\int_0^1 F'(\theta w + (1-\theta)y)d\theta(w - y) - \int_0^1 F'(\theta x + (1-\theta)z)d\theta(x - z)| \\
&\leq |\int_0^1 F'(\theta w + (1-\theta)y)d\theta||w - x - y + z| \\
&+ |\int_0^1 [F'(\theta x + (1-\theta)z) - F'(\theta w + (1-\theta)y)]d\theta|x - z| \\
&\leq \|F'\|_{L_\infty(\mathbb{R}^n)} |w - x - y + z| + Lip(F')(|w - x - y + z| + |y - z|)|x - z| \ .
\end{aligned}$$

Step 2. F seen as a composition operator on $H^\sigma_\infty(\mu)$ may be estimated as follows. Since $|F(x)-F(y)| \le \|F'\|_{L_\infty(\mathbb{R})}|x-y|$, it follows readily from the proofs of Proposition 2.3.2 and Proposition 3.3.1 in [9] that

$$\mathcal{E}^{(\sigma)}(F(u))^{1/2} \le \|F'\|_{L_\infty(\mathbb{R})}\,\mathcal{E}^{(\sigma)}(u)^{1/2}\ ,$$

$u \in H^\sigma_\infty(\mu)$, and by elementary arguments then also $\|F(u)\|_{\sigma,\infty} \le c\,\|u\|_{\sigma,\infty}$, i.e. F is a bounded operator. We turn to the estimate (i). Using the integral representation (4) of the transition operators $T^{(\sigma)}(t)$ the definition of the Dirichlet form $\mathcal{E}^{(\sigma)}$ may be rewritten as

$$\mathcal{E}^{(\sigma)}(u) = \sup_{t>0} E^{(\sigma)}_t(u),$$

where

$$E^{(\sigma)}_t(u) = \frac{1}{2t}\int_X\int_X (u(x)-u(y))^2 P^{(\sigma)}_t(x,dy)\mu(dx)$$

for $u \in L_2(\mu)$. If the desired estimate is provided for $E^{(\sigma)}_t$ with $t>0$ fixed, it carries over to $\mathcal{E}^{(\sigma)}$ by (17). To simplify notation we write E and $P(\cdot,dy)$ for $E^{(\sigma)}_t$ and $\frac{1}{2t}P^{(\sigma)}_t(\cdot,dy)$ with $t>0$ fixed, respectively, such that

$$E(u) = \int_X\int_X (u(x)-u(y))^2 P(x,dy)\mu(dx).$$

As $t>0$,

$$E(u) \le c_t\,\|u\|_0^2 \tag{25}$$

with $0 < c_t < \infty$ by the Markov property.
An arbitrary function in $L_2(\mu)\cap L_\infty(\mu)$ can be approximated by a sequence of real valued step functions simultaneously in $L_2(\mu)$ and $L_\infty(\mu)$. By (25) this approximation property holds also with respect to E. Hence it suffices for our purposes to consider step functions

$$u = \sum_{i=1}^n a_i \mathbf{1}_{A_i} \quad \text{and} \quad v = \sum_{i=1}^n b_i \mathbf{1}_{B_i} \tag{26}$$

with $A_i, B_i \in \mathcal{X}$ forming partitions of X and $a_i, b_i \in \mathbb{R}$, $i = 1,...,n$. We have

$$E(u-v) = \sum_{ijkl}(a_i - b_k - a_j + b_l)\int_{A_j\cap B_l} P(x, A_i\cap B_k)\mu(dx)\ ,$$

and by the estimates on F from Step 1,

$$E(F\circ u-F\circ v)=\sum_{ijkl}(F(a_i)-F(b_k)-F(a_j)+F(b_l))^2\int_{A_i\cap B_k}\int_{A_j\cap B_l}P(x,dy)\mu(dx)$$

$$\leq c\,\|F'\|^2_{L_\infty(\mathbb{R})}\sum_{ikjl}(a_i-b_k-a_j+b_l)^2\int_{A_i\cap B_k}\int_{A_j\cap B_l}P(x,dy)\mu(dx)$$

$$+\,c\,Lip(F')^2\left(\sum_{ikjl}(a_i-b_k-a_j+b_l)^2(b_k-b_l)^2\int_{A_i\cap B_k}\int_{A_j\cap B_l}P(x,dy)\mu(dx)\right.$$

$$\left.+\,c\sum_{ikjl}(a_j-b_l)^2(b_k-b_l)^2\int_{A_i\cap B_k}\int_{A_j\cap B_l}P(x,dy)\mu(dx)\right).$$

The first summand equals

$$c\,\|F'\|^2_{L_\infty(\mathbb{R})}\,E(u-v),$$

the second does not exceed

$$c\,Lip(F')^2\left(E(u-v)+\|u-v\|_0^2\right)\|v\|^2_{L_\infty(\mu)}.$$

Note that as $P(\cdot,dy)$ is Markovian,

$$\sum_{ikjl}(a_j-b_l)^2\int_{A_i\cap B_k}\int_{A_j\cap B_l}P(x,dy)\mu(dx)\leq\sum_{jl}(a_j-b_l)^2\mu(A_j\cap B_l)^2=\|u-v\|_0^2.$$

Altogether, we obtain a bound

$$E(F\circ u-F\circ v)\leq c\left(E(u-v)+\|u-v\|_0^2\right)\|v\|^2_{L_\infty(\mu)}\ .$$

By (25) the same bound holds for general $u,v\in L_2(\mu)$, and by the preceding discussion it provides the desired estimate on $\mathcal{E}^{(\sigma)}$. The remaining bounds $\|F(u)-F(v)\|_0\leq\|F'\|_{L_\infty(\mathbb{R})}\|u-v\|_0$ and $\|F(u)-F(v)\|_{L_\infty(\mu)}\leq\|F'\|_{L_\infty(\mathbb{R})}\|u-v\|_{L_\infty(\mu)}$ are easily seen, and (i) follows.

Step 3. Aiming at (ii) we establish elementary bounds also on the function G. By the mean value theorem,

$$|G(w)-G(x)-G(y)+G(z)-G(w')+G(x')+G(y')-G(z')|$$

$$=|\int_0^1[G'(\theta w+(1-\theta)y)-G'(\theta x+(1-\theta)z)]d\theta(w-y)$$

$$+\int_0^1 G'(\theta x+(1-\theta)z)d\theta(w-y-x+z)$$

$$-\int_0^1[G'(\theta w'+(1-\theta)y')-G'(\theta x'+(1-\theta)z')]d\theta(w'-y')$$

$$-\int_0^1 G'(\theta x'+(1-\theta)z')d\theta(w'-y'-x'+z')|$$

is seen to be bounded by

$$
\begin{aligned}
&|\int_0^1 [G'(\theta w + (1-\theta)y) - G'(\theta x + (1-\theta)z) - G'(\theta w' + (1-\theta)y') \\
&\qquad + G'(\theta x' + (1-\theta)z')]d\theta(w-y)| \\
&+ |\int_0^1 [G'(\theta w' + (1-\theta)y') - G'(\theta x' + (1-\theta)z')](w - y - w' + y')| \\
&+ |\int_0^1 [G'(\theta x + (1-\theta)z) - G'(\theta x' + (1-\theta)z')](w - x - y + z)| \\
&+ |\int_0^1 G'(\theta x' + (1-\theta)z')d\theta(w - x - y + z - w' + x' + y' - z')| \\
&=: S_1 + S_2 + S_3 + S_4 \ .
\end{aligned}
$$

Further applications of the mean value theorem yield

$$
\begin{aligned}
S_1 \le\ & Lip(G'')(|w - x - y + z - w' + x' + y' - z'| + |y - z - y' + z'| \\
&+ |x - z - x' + z'| + |z - z'|)(|w' - x' - y' + z'| + |y' - z'|)|w - y| \\
+ \|G''\|_{L_\infty(\mathbb{R})}\ & (|w - x - y + z - w' + x' + y' - z'| + |y - z - y' + z'|)|w - y|
\end{aligned}
$$

and

$$
S_2 \le \|G''\|_{L_\infty(\mathbb{R})} (|w' - y' - x' + z'| + |y' - z'|)|w - y - w' + y'| \ ,
$$

as well as

$$
S_3 \le \|G''\|_{L_\infty(\mathbb{R})} (|x - x' - z + z'| + |z - z'|)|w - x - y + z|
$$

and

$$
S_4 \le \|G'\|_{L_\infty(\mathbb{R}^n)} |w - x - y + z - w' + x' + y' - z'| \ .
$$

Step 4. Now the composition operator G on $H^\sigma_\infty(\mu)$ may be bounded similarly as F in Step 2. In addition to $u_1 := u$ and $v_1 := v$ from (26) consider two step functions

$$
u_2 = \sum_{i=1}^n a'_i \mathbf{1}_{A'_i} \quad \text{and} \quad v_2 = \sum_{i=1}^n b'_i \mathbf{1}_{B'_i} \ ,
$$

where A'_i, B'_i, α'_i and β'_i satisfy the same hypotheses as A_i etc. above. We have

$$
\begin{aligned}
E(u_1 - v_1 - u_2 + v_2) = \sum_{ikjli'k'j'l'} & (a_i - b_k - a_j + b_l - a'_{i'} + b'_{k'} + a'_{j'} - b'_{l'})^2 \times \\
& \times \int_{A_i \cap B_k \cap A'_{i'} \cap B'_{k'}} \int_{A_j \cap B_l \cap A'_{j'} \cap B'_{l'}} P(x, dy)\mu(dx)
\end{aligned}
$$

and

$$
\begin{aligned}
&E(G\circ u_1 - G\circ v_1 - G\circ u_2 + G\circ v_2)\\
&= \sum_{ikjli'k'j'l'} (G(a_i) - G(b_k) - G(a_j) + G(b_l) - G(a'_{i'}) + G(b'_{k'}) + G(a'_{j'}) - G(b'_{l'}))^2 \times\\
&\qquad\qquad \times \int_{A_i\cap B_k\cap A'_{i'}\cap B'_{k'}} \int_{A_j\cap B_l\cap A'_{j'}\cap B'_{l'}} P(x,dy)\mu(dx)\\
&\le \sum_{ikjli'k'j'l'} (S_1+S_2+S_3+S_4)^2 \int_{A_i\cap B_k\cap A'_{i'}\cap B'_{k'}} \int_{A_j\cap B_l\cap A'_{j'}\cap B'_{l'}} P(x,dy)\mu(dx)
\end{aligned}
$$

where the summands $S_1, ..., S_4$ are defined as in Step 3, but now with $w = a_i$, $x = b_k$, $y = a'_{i'}$, $z = b'_{k'}$ and $w' = a_j$, $x' = b_l$, $y' = a'_{j'}$, $z' = b'_{l'}$. Using the previous estimates from Step 3 and discarding some of the summands by the hypotheses $\|u_i\|_{\sigma,\infty} \le 1$, $\|v_i\|_{\sigma,\infty} \le 1$, $i = 1, 2$, we observe

$$
\begin{aligned}
&\sum_{ikjli'k'j'l'} S_1^2 \int_{A_i\cap B_k\cap A'_{i'}\cap B'_{k'}} \int_{A_j\cap B_l\cap A'_{j'}\cap B'_{l'}} P(x,dy)\mu(dx)\\
&\le Lip(G'')^2 \Bigg(c \sum_{ikjli'k'j'l'} (a_j - b_l - a'_{j'} + b'_{l'})^2 \int\int_{\dots} P(x,dy)\mu(dx)\\
&\quad + c \sum_{ikjli'k'j'l'} (a'_{j'} - b'_{l'})^2 \int\int_{\dots} P(x,dy)\mu(dx)\Bigg)\\
&\quad + \|G''\|_{L_\infty(\mathbb{R})} \Bigg(c \sum_{ikjli'k'j'l'} (a_i - b_k - a'_{i'} + b'_{k'} - a_j - b_l - a'_{j'} + b'_{l'})^2 \int\int_{\dots} P(x,dy)\mu(dx)\\
&\qquad + c \sum_{ikjli'k'j'l'} (\alpha'_{i'} - \beta'_{k'} - \alpha'_{j'} + \beta'_{l'})^2 \int\int_{\dots} P(x,dy)\mu(dx)\Bigg)\\
&\quad \le c\, Lip(G'')^2 \left(\|u_1 - u_2 - v_1 + v_2\|_0^2 + \|u_2 - v_2\|_0^2\right)\\
&\qquad + c\,\|G''\|_{L_\infty(\mathbb{R})} \left(E(u_1 - v_1 - u_2 + v_2) + E(u_2 - v_2)\right),
\end{aligned}
$$

where the dots stand for the domain of integration $(A_i \cap B_k \cap A'_{i'} \cap B'_{k'}) \times (A_j \cap B_l \cap A'_{j'} \cap B'_{l'})$. For the summands involving S_2, S_3 and S_4 we similarly obtain the upper bounds

$$
c\,\|G''\|_{L_\infty(\mathbb{R})} \left(\|u_1 - u_2 - v_1 + v_2\|_0^2 + \|u_2 - v_2\|_0^2\right),
$$

$$
c\,\|G''\|_{L_\infty(\mathbb{R})} \|u_1 - u_2 - v_1 + v_2\|_0^2
$$

and

$$
c\,\|G''\|_{L_\infty(\mathbb{R})} E(u_1 - v_1 - u_2 + v_2),
$$

respectively. Clipping these estimates, a suitable estimate for $\mathcal{E}^{(\sigma)}$ follows as before. For the $\|\cdot\|_0$- and $\|\cdot\|_{L_\infty(\mu)}$-parts of the norm bounds may be deduced using the estimates in Step 1 with G in place of F. □

4. Refined multiplications results

Under additional assumptions the product estimate (20) can be refined. As a consequence we can then avoid a factor involving the spectral dimension in (24).
The following implication provides a regularity result for the measure μ which will be useful in the proof of Proposition 4.1 below.

Lemma 4.1. *Assume that (MMS) and (HKE(β)) hold. Then μ is a d_f-*measure*: There is some $C>0$ such that*

$$C^{-1}\varrho^{d_f} \leq \mu(B(x,\varrho)) \leq C\varrho^{d_f} \tag{27}$$

for $0<\varrho<R_0$.

For a proof see [24], Proposition 2.6.
We may now pass to an $L_p(\mu)$-setting so that $L_\infty(\mu)$ is no longer needed.

Proposition 4.1. *Let $0<\beta<\delta<\frac{d_S}{2}\wedge 1$ and $p=\frac{d_S}{d_S-\delta}$. Assume (MMS) and (HKE(β)) hold and the semigroup is ultracontractive with spectral dimension d_S. Then we have*

$$\|uv\|_{H_p^\beta(\mu)} \leq c\ \|u\|_\delta \, \|v\|_\delta$$

for any $u,v\in H^\delta(\mu)$.

Proof. As in [25], Section 5 we observe that according to (10)

$$(A+I)^{\beta/2}\widetilde{u} = (J^\beta(\mu))^{-1}\widetilde{u} = c\int_0^\infty t^{-\beta/2-1}(\widetilde{u}-e^{-t}T(t)\widetilde{u})dt$$

for any $\widetilde{u}\in H_p^\beta(\mu)$. Therefore

$$\|\widetilde{u}\|_{H_p^\beta(\mu)} \leq c\,\|\widetilde{u}\|_{L_p(\mu)} + c\,\|Q_1(\widetilde{u})\|_{L_p(\mu)}\ ,$$

where

$$Q_1\widetilde{u} = \int_0^\infty e^{-t}t^{-\beta/2-1}(\widetilde{u}-T(t)\widetilde{u})dt\ .$$

Plugging in the transition densities,

$$|Q_1\widetilde{u}(x)| \leq \int_X |\widetilde{u}(x)-\widetilde{u}(y)|j^{(\beta)}(x,y)\mu(dy)$$

for μ-a.a. $x\in X$, where

$$j^{(\beta)}(x,y) = \int_0^\infty e^{-t}t^{-\beta/2-1}p(t,x,y)dt \leq cd(x,y)^{-d_f-\beta w/2}, \tag{28}$$

since by (HKE(β)) and (21),

$$\begin{aligned}\int_0^\infty e^{-t}t^{-\beta/2-1}p(t,x,y)dt &\leq c\int_0^{R_0} e^{-t}t^{-\beta/2-d_f/w-1}\Phi_2(t^{-1/w}d(x,y))dt\\ &\quad + c\int_{R_0}^\infty e^{(\omega-1)t}dt\\ &\leq c\,d(x,y)^{-d_f-\beta w/2}\int_0^{R_0} s^{d_f+\beta w/2-1}\Phi_2(s)ds\end{aligned}$$

for $x, y \in X$ with $0 < d(x,y) \le R_0$, the last integral being finite.
Given $u, v \in H^\delta(\mu)$ we now consider the pointwise product uv in place of $\widetilde{u}$. First note that

$$\begin{aligned}\|Q_1(uv)\|_{L_p(\mu)} &\le \left(\int_X \left(\int_X |u(x)v(x) - u(y)v(y)| j^{(\beta)}(x,y)\mu(dy)\right)^p \mu(dx)\right)^{1/p} \\ &\le \left(\int_X |u(x)|^p \left(\int_X |v(x) - v(y)| j^{(\beta)}(x,y)\mu(dy)\right)^p \mu(dx)\right)^{1/p} \\ &\quad + \left(\int_X \left(\int_X |v(y)||u(x) - u(y)| j^{(\beta)}(x,y)\mu(dy)\right)^p \mu(dx)\right)^{1/p} \\ &=: I_1 + I_2 \ .\end{aligned}$$

By Hölder's inequality
(29)

$$I_1 \le \left(\int_X |u|^{pq'} \mu(dx)\right)^{1/(pq')} \left(\int_X \left(\int_X |v(x) - v(y)| j^{(\beta)}(x,y)\mu(dy)\right)^{pq} \mu(dx)\right)^{1/(pq)} ,$$

where $q > 1$ is chosen such that $pq = 2$ and q' is its Hölder conjugate, $\frac{1}{q} + \frac{1}{q'} = 1$. Note that by hypothesis $p < 2$. Now put $r := pq'$. Then $r = \frac{2p}{2-p}$ and with our initial choice of p this yields $r = \frac{2d_f}{d_f - \delta w}$. By [24], Theorem 4.1 the first factor in (29) therefore satisfies the Sobolev inequality

$$\|u\|_{L_r(\mu)} \le c \, \|u\|_\delta \ .$$

Consider the second factor in (29). For $\varepsilon > 0$ and $x \in X$ fixed, define the measure

$$\widetilde{\mu}_x(dy) := \mathbf{1}_{B(x,1)}(y) \, \frac{1}{d(x,y)^{d_f - \varepsilon}} \mu(dy) \ .$$

Since μ is a d_f-measure, we have

$$\widetilde{\mu}_x(X) = \int_{d(x,y)<1} d(x,y)^{-(d_f - \varepsilon)} \mu(dy) \le c \int_0^1 \varrho^{\varepsilon - d_f} m_x(d\varrho) \le C \tag{30}$$

where $m_x((0,s)) = \mu(B(x,s))$ and the bound is uniform in $x \in X$. Hence for any x, $\widetilde{\mu}_x$ is a finite measure on X and by Hölder's inequality w.r.t. it,

$$\begin{aligned}&\int_X \left(\int_{d(x,y)<1} |u(x) - v(y)| j^{(\beta)}(x,y)\mu(dy)\right)^2 \mu(dx) \qquad (31)\\ &\le \int_X \widetilde{\mu}_x(X)^2 \int_{d(x,y)<1} (v(x) - v(y))^2 (j^{(\beta)}(x,y))^2 d(x,y)^{2(d_f - \varepsilon)} \widetilde{\mu}_x(dy)\mu(dx) \ .\end{aligned}$$

We have

$$(j^{(\beta)}(x,y))^2 d(x,y)^{d_f - \varepsilon} \le c \, \frac{d(x,y)^{d_f - \varepsilon}}{d(x,y)^{2d_f + 2\beta w}} = \frac{c}{d(x,y)^{d_f + 2\beta w + \varepsilon}} \ ,$$

and for the choice $\varepsilon := 2(\delta - \beta)w$ (31) admits the bound

$$c \int_X \int_X \frac{(v(x) - v(y))^2}{d(x,y)^{d_f + 2\delta w}} \mu(dy)\mu(dx) \le c \, \|v\|_\delta^2 \ ,$$

see Corollary 3.4 in [24]. By (HKE(β)), the triangle inequality, and the symmetry of the kernel (28), we find

$$\left(\int_X \left(\int_{d(x,y)\geq 1} |v(x)-v(y)| j^{(\beta)}(x,y)\mu(dy)\right)^2 \mu(dx)\right)^{1/2} \leq c\,\|v\|_0 \ .$$

Altogether, $I_1 \leq c\,\|u\|_\delta\,\|v\|_\delta$. To estimate I_2, use the triangle inequality,

$$I_2 \leq \int_X |v(y)| \left(\int_X |u(x)-u(y)|^p (j^{(\beta)}(x,y))^p \mu(dx)\right)^{1/p} \mu(dy) \ .$$

Hölder's inequality with respect to $\widetilde{\mu}_y$, constructed similarly as before, yields

$$\int_{d(x,y)<1} |u(x)-u(y)|^p (j^{(\beta)}(x,y))^p \mu(dx)$$
$$\leq \left(\int_{d(x,y)<1} \widetilde{\mu}_y(dx)\right)^{(2-p)/2} \left(\int_{d(x,y)<1} |u(x)-u(y)|^2 (j^{(\beta)}(x,y))^2 d(x,y)^{2(d_f-\varepsilon)} \widetilde{\mu}_y(dx)\right)^{1/2}$$

with the first integral bounded uniformly in $y \in X$. By Cauchy-Schwarz and the previous arguments,

$$I_2 \leq \left(\int_X |v(y)|^2 \mu(dy)\right)^{1/2} \times$$
$$\times \left\{ \left(\int_X \int_X \frac{|u(x)-u(y)|^2}{d(x,y)^{d_f+2\delta w}} \mu(dx)\mu(dy)\right)^{1/2} + \left(\int_X |u(x)|^2 \mu(dx)\right)^{1/2} \right\}$$
$$\leq c\,\|u\|_\delta\,\|v\|_\delta \ ,$$

cf. [24], Theorem 3.3. It remains to estimate $\|uv\|_{L_p(\mu)}$. This is similar, but simpler, we omit it. □

As before, define the product of a function $u \in H^\delta(\mu)$ and a dual element $H_q^{-\beta}(\mu)$ by

$$(v, uz) := (vu, z)\ , v \in H^\delta(\mu) \ .$$

Here q is the Hölder conjugate of p above. Proposition 4.1 yields a bare-hands version of a multiplication result used in [22]:

Corollary 4.1. *Let $0 < \beta < \delta < \frac{d_S}{2} \wedge 1$ and $q = \frac{d_S}{\delta}$. Assume (MMS) and (HKE(β)) hold and the semigroup is ultracontractive. Then*

$$\|uz\|_{-\beta} \leq c\ \|u\|_\delta\,\|z\|_{H_q^{-\beta}(\mu)}$$

for any $u \in H^\delta(\mu)$ and $z \in H_q^{-\beta}(\mu)$.

Proof. Let $v \in H^\beta(\mu)$. Then by duality and Proposition 4.1,

$$|(v,uz)| = |(vu,z)| \leq \|vu\|_{H_p^\beta(\mu)}\,\|z\|_{H_q^{-\beta}(\mu)} \leq \|v\|_\delta\,\|u\|_\delta\,\|z\|_{H_q^{-\beta}(\mu)} \ ,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. □

Defining $T(t)z$, $z \in H_q^{-\beta}(\mu)$, $t \in (0,t_0)$, similarly as in (19), we now observe

$$\|T(t)z\|_0 \le ct^{-\beta/2}\,\|z\|_{H_q^{-\beta}(\mu)}\ . \tag{32}$$

5. The integral operator

Recall problem (1), the definition (2) of a mild solution and the hypotheses of Theorem 1.1 *(Case I)* with $\theta = 1$, say. Assume that $0 < \beta \le \delta < 1$ and $u(s) \in H^\delta_\infty(\mu)$ for $s \in (0,t)$. Then the Markov property of $\mathcal{E}^{(\delta)}$ and (20) tell that for $w \in H^{-\beta}(\mu)$, $G(u(s))w \in (H^\beta_\infty(\mu))^*$. Taking into account the mapping properties (12) and (24) of $(T(t))_{t\ge 0}$,

$$U(t;s)w := T(t-s)G(u(s))w\ ,\ w \in H^{-\beta}(\mu),$$

is seen to define an element $U(t;s)$ of $L(H^{-\beta}(\mu), H^\delta_\infty(\mu))$. The semigroup operator is applied to the product $G(u(s))w$.
For *Case II* we may use Corollary 4.1 and (32) and consider $w \in H_q^{-\beta}(\mu)$, $0 < \beta < \delta < d_s/2$, $q = \frac{d_S}{\delta}$, for *Case III* with linear G it also suffices to have $u(s) \in H^\delta(\mu)$.
Now interpret $U(t;s)$ as an operator-valued function of $s \in (0,t)$, where t is fixed. If $s \mapsto u(s)$ is sufficiently regular, $s \mapsto U(t;s)$ inherits this regularity and we may consider its *left-sided Weyl-Marchaud fractional derivative of order* η, defined by

$$D^\eta_{0+}U(t;s) := \frac{\mathbf{1}_{(0,t)}(s)}{\Gamma(1-\eta)}\left(\frac{U(t;s)}{s^\eta} + \eta\int_0^s \frac{U(t;s)-U(t;\tau)}{(s-\tau)^{\eta+1}}d\tau\right).$$

Given $\theta \in (0,1)$, we can similarly define $U^{(\theta)}(t;s)$ with fractional derivative $D^\eta_{0+}U^{(\theta)}(t;\cdot)$. For a sufficiently nice function z on $[0,t]$,

$$D^{1-\eta}_{t-}z_t(s) := \frac{(-1)^{1-\eta}\mathbf{1}_{(0,t)}(s)}{\Gamma(\eta)}\left(\frac{z(s)-z(t)}{(t-s)^{1-\eta}} + (1-\eta)\int_s^t \frac{z(s)-z(\tau)}{(\tau-s)^{(1-\eta)+1}}d\tau\right)\ ,$$

is the *right-sided Weyl-Marchaud fractional derivative of order* $1-\eta$ of its regulated version $z_t(s) := \mathbf{1}_{(0,t)}(s)(z(s)-z(t))$. For these definitions and some background see [34], [45], [46].
We define our integral operator in (2) as follows:

Definition 5.1. Given $\theta \in (0,1]$, $t \in [0,t_0]$, $0 < \eta < 1/2$ and sufficiently regular functions u and z on $[0,t]$, put

$$\int_0^t T^{(\theta)}(t-s)G(u(s))dz(s) := (-1)^\eta \int_0^t D^\eta_{0+}U^{(\theta)}(t;s)D^{1-\eta}_{t-}z_t(s)ds\ . \tag{33}$$

To obtain the later results it is necessary to have $\eta < 1/2$ anyway and imposing this condition already now, we avoid a discussion of boundary correction terms.
This definition is a version of [21], Definition 3.1. We may state similar conditions as there to ensure the existence of (33):

Lemma 5.1. *Let θ, t and η be as in Definition 5.1. Assume u is such that $D^{\eta}_{0+}U^{(\theta)}(t;\cdot) \in L_1([0,t], L(H^{-\beta\theta}(\mu), H^{\delta\theta}_{\infty}(\mu)))$ and z is such that $D^{1-\eta}_{t-}z_t \in L_{\infty}([0,t], H^{-\beta\theta}(\mu))$, where $0 < \beta \le \delta < 1$* (Case I). *Then the right-hand side of (33) exists as an element of $H^{\delta\theta}_{\infty}(\mu)$ and is independent of the particular choice of η.*
Similarly for $H^{\beta\theta}_{q}(\mu)$, $0 < \beta < \delta < \frac{d_S}{2} \wedge 1$, $q = \frac{d_S}{\delta}$, in place of $H^{-\beta\theta}(\mu)$ (Case II) *and also if in addition $H^{\delta\theta}(\mu)$ replaces $H^{\delta\theta}_{\infty}(\mu)$* (Case III).

The existence is easy to see, the independence of the choice of η is a consequence of integration by parts formulae. The arguments are the same as for the scalar version of (33), cf. [45].

In the following we verify the hypotheses of Lemma 5.1 in the respective cases and show how $D^{\eta}_{0+}U^{(\theta)}(t;\cdot)$ can be rewritten in terms of the semigroup and fractional powers of its generator.
Given $0 < \eta < 1$ and $\sigma \ge 0$, let $W^{\eta}([0,t_0], H^{\sigma}_{\infty}(\mu))$ denote the space of $H^{\sigma}_{\infty}(\mu)$-valued functions v on $[0,t_0]$ such that

$$\|v\|_{W^{\eta}([0,t_0],H^{\sigma}_{\infty}(\mu))} := \sup_{0\le t\le t_0}\left(\|v(t)\|_{\sigma,\infty} + \int_0^t \frac{\|v(t)-v(\tau)\|_{\sigma,\infty}}{(t-\tau)^{\eta+1}}d\tau\right) < \infty\ .$$

For $0 < \eta < 1$ and $0 < \sigma \le 1$ let $C^{\eta}([0,t_0], H^{-\sigma}(\mu))$ denote the space of η-Hölder continuous $H^{-\sigma}(\mu)$-valued functions v on $[0,t_0]$ such that

$$\|v\|_{C^{\eta}([0,t_0],H^{-\sigma}(\mu))} := \sup_{0\le t\le t_0}\|v(t)\|_{-\sigma} + \sup_{0\le\tau<t\le t_0}\frac{\|v(t)-v(\tau)\|_{-\sigma}}{(t-\tau)^{\gamma}} < \infty\ .$$

By obvious modifications we can define similar spaces with $H^{\sigma}(\mu)$, $H^{-\sigma}(\mu)$, $H^{-\sigma}_{q}(\mu)$ or $(H^{\sigma}_{\infty}(\mu))^*$ in place of $H^{\sigma}_{\infty}(\mu)$ or $H^{-\sigma}(\mu)$.

Given $z \in C^{1-\alpha}([0,t_0], H^{-\beta\theta}(\mu))$ according to Theorem 1.1 *(Case I)* and η slightly bigger than α,

$$w(s) := D^{1-\eta}_{t-}z_t(s)\ ,\ s \in [0,t] \tag{34}$$

defines a function in $L_{\infty}([0,t], H^{-\beta\theta}(\mu))$. (In fact, the fractional derivative converges pointwise and w is still Hölder continuous.) For z as in Theorems 1.2 and 1.3 *(Cases II and III)*, we obtain $w \in L_{\infty}([0,t], H^{-\beta\theta}_{q}(\mu))$ with q according to Lemma 5.1.
The following is a version of Lemma B.1 from [22]. We formulate it for $\theta = 1$ only, the modifications needed for $\theta \in (0,1)$ are straightforward.

Lemma 5.2. *Suppose that zero is not an eigenvalue of A. Let $0 < \eta < 1/2$ and $t \in (0,t_0)$.*

(i) *If $0 < \beta \le \delta < 1$, $u \in W^{\eta}([0,t], H^{\delta}_{\infty}(\mu))$ and*

$$\delta \vee \frac{d_S}{2} < 2 - 2\eta - (\beta \vee \frac{d_S}{2}),$$

then $D^{\eta}_{0+}U(t;\cdot)$ *converges in* $L_1([0,t], L(H^{-\beta}(\mu), H^{\delta}_{\infty}(\mu)))$ *and*

$$(35)\quad D^{\eta}_{0+}U(t;s) = D^{\eta}_{0+}(T(t-\cdot)G(u(\cdot))(s)$$
$$= \mathbf{1}_{(0,t)}(s)\Big\{-A^{\eta}T(t-s)G(u(s)) + c_{\eta}T(t-s)\int_s^{\infty} r^{-\eta-1}T(r)G(u(s))dr$$
$$+c_{\eta}\int_0^s r^{-\eta-1}T(r+t-s)[G(u(s)) - G(u(s-r))]dr.\Big\}$$

Here $c_{\eta} = \eta\Gamma(1-\eta)^{-1} = -\Gamma(-\eta)^{-1}$ *and* G *is as in Theorem 1.1.*

(ii) *Let* G *be as in Theorem 1.2. If* $0 < \beta < \delta < \frac{d_S}{2} \wedge 1$, $q = \frac{d_S}{\delta}$, $u \in W^{\eta}([0,t], H^{\delta}_{\infty}(\mu))$ *and*

$$\frac{d_S}{2} < 2 - 2\eta - (\beta \vee \frac{d_S}{2}),$$

then $D^{\eta}_{0+}U(t;\cdot)$ *converges in* $L_1([0,t], L(H^{-\beta}_q(\mu), H^{\delta}_{\infty}(\mu)))$ *and (35) holds.*

(iii) *Let* G *be linear as in Theorem 1.3. If* $0 < \beta < \delta < \frac{d_S}{2} \wedge 1$, $q = \frac{d_S}{\delta}$, $u \in W^{\eta}([0,t], H^{\delta}(\mu))$ *and*

$$\delta < 2 - 2\eta - (\beta \vee \frac{d_S}{2}),$$

then $D^{\eta}_{0+}U(t;\cdot)$ *converges in* $L_1([0,t], L(H^{-\beta}_q(\mu), H^{\delta}(\mu)))$ *and (35) holds.*

In (i) and (ii) we need to consider bounded functions so that Proposition 3.1 yields key estimates. To obtain boundedness, we use the ultracontractivity of the semigroup, which requires the stronger conditions.
The rest of this section is concerned with the proof of Lemma 5.2. We comment on (i) only, because the modifications for (ii) and (iii) are obvious.

Proof. Given $t \in (0, t_0)$ and $\varepsilon > 0$, consider

$$\psi_{\varepsilon}(s) := \begin{cases} \int_0^{s-\varepsilon} \frac{T(t-s)G(u(s)) - T(t-\tau)G(u(\tau))}{(s-\tau)^{1+\eta}} d\tau \ , \ s \in (\varepsilon, t), \\ \frac{T(t-s)G(u(s))}{\eta}[\varepsilon^{-\eta} - s^{-\eta}] \ , \ s \in (0, \varepsilon], \end{cases}$$

and

$$\varphi_{\varepsilon}(s) := \frac{c_{\eta}}{\eta}\frac{T(t-s)G(u(s))}{s^{\eta}} + c_{\eta}\psi_{\varepsilon}(s) \ .$$

Let $\varphi(s)$ denote the right-hand side of (35). We show that

$$(36)\qquad \lim_{\varepsilon\to 0}\int_0^t \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \|\varphi_{\varepsilon}(s)w - \varphi(s)w\|_{\delta,\infty}\, ds = 0 \ .$$

This can be seen following the estimates in the proof of Lemma B.1 in [22]: By a simple change of variables and the semigroup property,

$$
\begin{aligned}
(37)\quad \mathbf{1}_{(\varepsilon,t)}(s)\varphi_\varepsilon(s) &= c_\eta \int_\varepsilon^\infty \frac{[I-T(r)]T(t-s)G(u(s))}{r^{\eta+1}}dr \\
&- c_\eta \left(T(t-s)\int_s^\infty \frac{[I-T(r)]G(u(s))}{r^{\eta+1}}dr - \eta^{-1}\frac{T(t-s)G(u(s))}{s^\eta}\right) \\
&+ c_\eta T(t-s)\int_\varepsilon^s \frac{T(r)[G(u(s))-G(u(s-r))]}{r^{\eta+1}}dr.
\end{aligned}
$$

The $L_1([0,t], L(H^{-\beta}(\mu), H^\delta_\infty(\mu)))$-norm of the difference between the second term on the right-hand side of (37) and the second term on the right-hand side of (35) is

$$
\begin{aligned}
&\int_0^\varepsilon \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \left\| T(t-s)\left(\int_s^\infty r^{-\eta-1}T(r)G(u(s))dr\right) w\right\|_{\delta,\infty} ds \\
&= \int_0^\varepsilon \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \left\| T(t-s)\int_s^\infty r^{-\eta-1}T(r)G(u(s))w dr\right\|_{\delta,\infty} ds \\
&\le c\int_0^\varepsilon (t-s)^{-(\delta/2\vee d_S/4)-(\beta/2\vee d_S/4)}\int_s^\infty r^{-\eta-1}dr \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1}\|G(u(s))w\|_{(H^\beta_\infty(\mu))^*} ds
\end{aligned}
$$

(38)

$$
\le c\,\|u\|_{W^\eta([0,t],H^\delta_\infty(\mu))}\int_0^\varepsilon (t-s)^{-(\delta/2\vee d_S/4)-(\beta/2\vee d_S/4)}s^{-\eta}ds\ ,
$$

which tends to zero as ε does. We have used (12), (21), (24) and (20). In the first line we have read w as bounded linear operator acting on the term in brackets by pointwise evaluation at $w \in H^{-\beta}(\mu)$. By the basic rules of Bochner's integration we therefore observe the first identity.
For the difference between the last terms in (37) and (35) we may proceed similarly to observe the same effect. The difference between the first terms can be tackled as follows, cf. [22]:

$$
\begin{aligned}
(39)\quad A^\eta &\left[\int_0^\infty T(\varepsilon r)T(t-s)G(u(s))w\, q_\eta(r)dr\right] \\
&= \int_\varepsilon^\infty \frac{[I-T(r)]T(t-s)G(u(s))w}{r^{\eta+1}}dr\ ,\ w\in H^{-\beta}(\mu),
\end{aligned}
$$

is a special case of an identity shown in [7], Section 2. Here q_η is an integrable function on $(0,\infty)$, determined by its Laplace transform,

$$
\int_0^\infty e^{-\lambda r}q_\eta(r)dr = \lambda^{-\eta}\int_1^\infty \frac{1-e^{-\lambda r}}{r^{\eta+1}}dr\ ,\lambda>0\ .
$$

Its integral yields

$$
\int_0^\infty q_\eta(r)dr = \Gamma(-\eta)\ .
$$

Using (39),

$$\int_{\varepsilon}^{t} \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \Big\| -A^{\eta}T(t-s)G(u(s))w + \frac{1}{\Gamma(-\eta)}\int_{\varepsilon}^{\infty} \frac{[I-T(r)]T(t-s)G(u(s))w}{r^{\eta+1}}dr \Big\|_{\delta,\infty} ds \tag{40}$$

$$\le c\int_0^t\int_0^{\infty} \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \|A^{\eta}[T(\varepsilon r)-I]T(t-s)G(u(s))w\|_{\delta,\infty}\, q_{\eta}(r)drds \ .$$

An integrable majorant is provided by

$$\int_0^t (t-s)^{-\eta-(\delta/2\vee d_S/4)-(\beta/2\vee d_S/4)} \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \|G(u(s))w\|_{(H^{\beta}_{\infty}(\mu))^*}\int_0^{\infty} q_{\eta}(r)drds \ ,$$

we have used (12) and (24). On the other hand, we have

$$\sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \|[T(\varepsilon r)-I]A^{\eta}T(t-s)G(u(s))w\|_{\delta}$$

$$\le c(\varepsilon r)^{\kappa/2} \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \|A^{\eta}T(t-s)G(u(s))w\|_{\delta-\kappa}$$

$$\le c(\varepsilon r)^{\kappa/2}(t-s)^{-\eta-\delta/2-\kappa/2-(\beta/2\vee d_S/4)}\, \|u\|_{W^{\eta}([0,t],H^{\delta}_{\infty}(\mu))} \tag{41}$$

by (13) and similarly

$$\sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \|[T(\varepsilon r)-I]A^{\eta}T(t-s)G(u(s))w\|_{\infty}$$

$$\le c(t-s)^{-d_S/4} \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \Big\|[T(\varepsilon r)-I]A^{\eta}T(\frac{t-s}{2})G(u(s))w\Big\|_0$$

$$\le c(\varepsilon r)^{\kappa/2}(t-s)^{-d_S/4} \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \Big\|A^{\eta}T(\frac{t-s}{2})G(u(s))w\Big\|_{\kappa}$$

$$\le c(\varepsilon r)^{\kappa/2}(t-s)^{-\eta-d_S/4-\kappa/2-(\beta/2\vee d_S/4)}\, \|u\|_{W^{\eta}([0,t],H^{\delta}_{\infty}(\mu))} \tag{42}$$

for every fixed fixed $(s,r)\in(0,t)\times(0,\infty)$, where $\kappa>0$ is a small number. Hence by dominated convergence the expression (40) tends to zero as ε does. Finally,

$$\mathbf{1}_{(0,\varepsilon)}(s)\varphi_{\varepsilon}(s) = \frac{c_{\eta}}{\eta}\frac{T(t-s)G(u(s))}{\varepsilon^{\eta}}$$

and

$$\varepsilon^{-\eta}\int_0^{\varepsilon} \sup_{\|w\|_{H^{-\beta}(\mu)}\le 1} \|T(t-s)G(u(s))w\|_{\delta}\, ds$$

$$\le c\,\varepsilon^{-\eta}\, \|u\|_{W^{\gamma}([0,t],H^{\delta}_{\infty}(\mu))}\int_0^{\varepsilon}(t-s)^{-(\beta/2\vee d_S/4)-(\delta/2\vee d_S/4)}ds \ .$$

By the concavity of $r\mapsto r^{\sigma}$ for $\sigma\in(0,1)$, the last integral is bounded by

$$c\,\varepsilon^{1-(\beta/2\vee d_S/4)-(\delta/2\vee d_S/4)} \ ,$$

and since $\eta < 1 - (\beta/2 \vee d_S/4) - (\delta/2 \vee d_S/4)$, the last line above tends to zero as ε does. Clipping the estimates we obtain (36). $\square$

Note that $D^{\eta}_{0+} U(t;s)$ converges also pointwise for any fixed $s \in (0,t)$.

6. Cauchy problems on metric measure spaces

We now follow [22] to prove Theorems 1.1, 1.2 and 1.3. Only the case $\theta = 1$ will be discussed, the modifications needed for $\theta \in (0,1)$ are mentioned at the end of this section.
Using Lemma 5.2 on the semigroup $(e^{-\omega t} T(t))_{t\geq 0}$ with $\omega \geq 0$ large enough we obtain a formulation for (33) in terms of fractional powers $(\omega I + A)^{\eta}$:

$$\int_0^t T(t-s) G(u(s)) dz(s) =$$

(43)

$$\begin{aligned}
&(-1)^{\eta+1} \int_0^t (\omega I + A)^{\eta} T(t-s) \left[G(u(s)) D^{1-\eta}_{t-} z_t(s) \right] ds \\
&+ c_\eta (-1)^{\eta} \int_0^t \int_0^s (s-\tau)^{-\eta-1} T(t-\tau) \left[(G(u(s)) - G(u(\tau)) D^{1-\eta}_{t-} z_t(s) \right] d\tau ds \\
&+ c_\eta (-1)^{\eta} \int_0^t \int_s^{\infty} e^{-\omega\tau} \tau^{-\eta-1} T(\tau + t - s) \left[G(u(s)) D^{1-\eta}_{t-} z_t(s) \right] d\tau ds \\
&+ c_\eta (-1)^{\eta} \int_0^t \int_0^s \tau^{-\eta-1} (e^{-\omega\tau} - 1) T(\tau + t - s) \left[G(u(s)) D^{1-\eta}_{t-} z_t(s) \right] d\tau ds,
\end{aligned}$$

where $c_\eta = \eta \Gamma(1-\eta)^{-1}$ and the semigroup operators each apply to the whole term in square brackets. If zero is not an eigenvalue, we may use $\omega = 0$. In this case the last summand vanishes and we obtain the expression that had been used to define the integral operator in [22].

Next introduce auxiliary equivalent norms on the space $W^{\eta}([0,t_0], H^{\sigma}_{\infty}(\mu))$, $0 < \eta < 1$, $\sigma \geq 0$, by

$$\|v\|^{(\varrho)}_{W^{\eta}([0,t_0],H^{\sigma}_{\infty}(\mu))} := \sup_{0\leq t\leq t_0} e^{-\varrho t} \left(\|v(t)\|_{\sigma,\infty} + \int_0^t \frac{\|v(t) - v(\tau)\|_{\sigma,\infty}}{(t-\tau)^{\eta+1}} d\tau \right) < \infty \,,$$

where $\varrho \geq 1$ is a parameter, cf. [22], [31]. Similarly for one of the other potential spaces in place of $H^{\sigma}_{\infty}(\mu)$.
We prove the contractivity of the operator (43) under the respective hypotheses.

Case I: Nonlinear multiplicative noise terms, general measure spaces.

Proposition 6.1. *Assume* $0 < \alpha, \beta, \gamma, \delta < 1$, $\alpha < \gamma < 1 - \alpha$, $\delta \geq \beta$ *and*

$$2\gamma + (\delta \vee \frac{d_S}{2}) < 2 - 2\alpha - (\beta \vee \frac{d_S}{2}) \ .$$

Let $z \in C^{1-\alpha}([0,t_0], H^{-\beta}(\mu))$ and let G be as in Theorem 1.1. Suppose that $R > 0$ is given. Then

(44)

$$\left\| \int_0^{\cdot} T(\cdot - s)G(u(s))dz(s) \right\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))} \leq C(\varrho)(1 + \|u\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))}),$$

$u \in W^\gamma([0,t_0], H^\delta_\infty(\mu))$, where $C(\varrho) > 0$ tends to zero as ϱ goes to infinity. For sufficiently large $\varrho_0 \geq 1$ (43) maps the closed ball

$$B^{(\varrho_0)}(0,R) = \left\{ u \in W^\gamma([0,t_0], H^\delta_\infty(\mu)) : \|g\|^{(\varrho_0)}_{W^\eta([0,t_0],H^\varrho_\infty(\mu))} \leq R \right\}$$

into itself and for $\varrho \geq \varrho_0$ large enough,

$$\left\| \int_0^{\cdot} T(\cdot - s)G(u(s))dz(s) - \int_0^{\cdot} T(\cdot - s)G(v(s))dz(s) \right\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))}$$
$$\leq C(\varrho) \|u - v\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))},$$

$u, v \in B^{(\varrho_0)}(0,R)$.

Proof. Note that for fixed $s \in (0,t)$,

$$\left\| G(u(s)) D^{1-\eta}_{t-} z_t(s) \right\|_{(H^\beta_\infty(\mu))^*} \leq \|G(u(s))\|_{\delta,\infty} \left\| D^{1-\eta}_{t-} z_t(s) \right\|_{(H^\beta(\mu))^*}$$

(45)
$$\leq c \|u(s)\|_{\delta,\infty} \|z\|_{C^{1-\alpha}([0,t_0],(H^\beta(\mu))^*)},$$

provided $\delta \geq \beta$ and η is slightly bigger than α. We have used (20), Proposition 3.1 and elementary estimates for $D^{1-\alpha}_{t-}$, see for instance [34] or Step 2 in the proof of Proposition 7.2 in [22].

Using the semigroup property of $(T(t))_{t\geq 0}$ together with its mapping properties (12), (21) and the basic product estimate (24), we observe a bound of type

$$\left\| \int_0^t (\omega I + A)^\eta T(t-s)[G(u(s)) D^{1-\eta}_{t-} z_t(s)] ds \right\|^{(\varrho)}_{\delta,\infty}$$
$$\leq c e^{-\varrho t} \int_0^t (t-s)^{-\eta - (\delta/2 \vee d_S/4) - (\beta/2 \vee d_S/4)} \|u(s)\|_{\delta,\infty}$$
$$\leq c e^{-\varrho t} \int_0^t (t-s)^{-\eta - (\delta/2 \vee d_S/4) - (\beta/2 \vee d_S/4)} \|u(s)\|_{\delta,\infty}$$
$$\leq c \|u\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))} \int_0^t e^{-\varrho(t-s)} (t-s)^{-\eta - (\delta/2 \vee d_S/4) - (\beta/2 \vee d_S/4)} ds$$
$$\leq c \|u\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))} \varrho^{\eta + (\delta/2 \vee d_S/4) + (\beta/2 \vee d_S/4) - 1}$$

for the first summand in (43). Note that by hypothesis $\eta + (\frac{\delta}{2} \vee \frac{d_S}{4}) + (\frac{\beta}{2} \vee \frac{d_S}{4}) < 1$. To obtain the remaining estimates, one can now apply Proposition 3.1 (i) and follow the proof of Proposition 7.2 in [22] line by line. The use of Proposition 3.1 requires $u(s)$ to be $L_\infty(\mu)$-bounded for (almost) any s. The estimate (44) follows and implies the stated invariance of $B^{(\varrho_0)}(0,R)$.

In a similar manner the contractivity result follows, use Proposition 3.1 (i) and (ii). See Step 3 in the proof of Proposition 7.3 in [22]. □

Case II: Nonlinear multiplicative noise terms, metric measure spaces.

We can similarly prove contractivity under the hypotheses of Theorem 1.2. In this case we use the refined product estimates in Corollary 4.1 and their consequence (32) for related bounds on the semigroup. Apart from that we can proceed as in the proof of Proposition 6.1.

Corollary 6.1. *Assume* $0 < \alpha, \beta, \gamma, \delta < 1$, $\alpha < \gamma < 1 - \alpha$, $0 < \beta < \delta < \frac{d_S}{2} \wedge 1$ *and*

$$2\gamma + \frac{d_S}{2} < 2 - 2\alpha - \beta \ .$$

Let $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$, $q = \frac{d_S}{\delta}$. *Then the assertions of Proposition 6.1 remain valid.*

Case III: Linear multiplicative noise terms, metric measure spaces.

Finally, a corresponding result holds true under the hypotheses of Theorem 1.3. In this case the integral operator (33) is linear, Proposition 3.1 and the $L_\infty(\mu)$-boundedness of $u(s)$ are not needed. The proof is the same as that of [22], Proposition 7.2.

Corollary 6.2. *Assume* $0 < \alpha, \beta, \gamma, \delta < 1$, $\alpha < \gamma < 1 - \alpha$, $0 < \beta < \delta < \frac{d_S}{2} \wedge 1$ *and*

$$2\gamma + \delta < 2 - 2\alpha - \beta \ .$$

Let $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$, $q = \frac{d_S}{\delta}$. *Then (33) is a bounded linear operator from* $u \in H^\delta(\mu)$ *to itself with*

$$\left\| \int_0^\cdot T(\cdot - s) u(s) dz(s) \right\|_{W^\gamma([0,t_0],H^\delta(\mu))}^{(\varrho)} \leq C(\varrho) \left\| u \right\|_{W^\gamma([0,t_0],H^\delta(\mu))} \ ,$$

$u \in H^\delta(\mu)$, *where* $C(\varrho) > 0$ *tends to zero as* ϱ *goes to infinity.*

The nonlinear term involving F can be treated similarly:

Lemma 6.1. *Assume* $0 < \gamma, \delta < 1$, $2\gamma + (\delta \vee \frac{d_S}{2}) < 2$ *and* F *is as in Theorem 1.1. Let* $R > 0$ *be given. Then*

$$\left\| \int_0^\cdot T(\cdot - s) F(u(s)) ds \right\|_{W^\gamma([0,t_0],H^\delta_\infty(\mu))}^{(\varrho)} \leq C(\varrho)(1 + \left\| u \right\|_{W^\gamma([0,t_0],H^\delta_\infty(\mu))}^{(\varrho)}) \tag{46}$$

$u \in W^\gamma([0, t_0], H^\delta_\infty(\mu))$, *where* $C(\varrho) > 0$ *tends to zero as* ϱ *goes to infinity. For* $\varrho_0 \geq 1$ *large enough,*

$$u \mapsto \int_0^\cdot T(\cdot - s) F(u(s)) ds$$

maps the closed ball $B^{(\varrho_0)}(0,R)$ into itself and for $\varrho \geq \varrho_0$ sufficiently large,

$$\left\| \int_0^{\cdot} T(\cdot - s)F(u(s))ds - \int_0^{\cdot} T(\cdot - s)F(v(s))ds \right\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))} \leq C(\varrho) \left\| u - v \right\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))} ,$$

$u, v \in B^{(\varrho_0)}(0,R)$.

The proof is the same as that of Lemma 7.4 in [22].

For $\theta = 1$ Theorem 1.1 and Corollary 1.1 now follow from Proposition 6.1, Lemma 6.1 and Banach's contraction principle: What concerns the initial condition f, note that by (13), (21) and (24),

$$\int_0^t \frac{\left\| T(t)f - T(\tau)f \right\|_{L_\infty(\mu)}}{(t-\tau)^{\gamma+1}} d\tau \leq c \left\| f \right\|_{2\gamma+\varepsilon} \int_0^t \tau^{-d_S/4} (t-\tau)^{\varepsilon-1} d\tau$$

and

$$\int_0^t \frac{\left\| T(t)f - T(\tau)f \right\|_\delta}{(t-\tau)^{\gamma+1}} d\tau \leq c \left\| f \right\|_{2\gamma+\delta+\varepsilon} \int_0^t (t-\tau)^{\varepsilon-1} d\tau ,$$

hence

$$\left\| T(\cdot)f \right\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))} \leq r$$

for some $r > 0$ and any $\varrho \geq 1$. We may now follow [31]: Let $\Phi^f_t(u)$ denote the right hand side of (2) with $\theta = 1$. By (44) and (46),

$$\left\| \Phi^f_\cdot(u) \right\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))} \leq r + C(\varrho)(1 + \left\| u \right\|^{(\varrho)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))}),$$

with $C(\varrho)$ going to zero for increasing ϱ. For $C(\varrho_0) \leq \frac{r}{1+2r}$ and $\left\| u \right\|^{(\varrho_0)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))} \leq 2r$ it follows that

$$\left\| \Phi^f_\cdot(u) \right\|^{(\varrho_0)}_{W^\gamma([0,t_0],H^\delta_\infty(\mu))} \leq 2r ,$$

in other words, $\Phi^f_\cdot$ maps $B^{(\varrho_0)}(0,R)$, $R := 2r$, into itself. Proposition 6.1 and Lemma 6.1 show its contractivity on $B^{(\varrho_0)}(0,R)$.

The results for $0 < \theta < 1$ may be obtained by considering A^θ resp. $T^{(\theta)}$ in place of A resp. T, note that $\mathcal{D}((I + A^\theta)^\alpha) = \mathcal{D}(A^{\alpha\theta}) = H^{2\alpha\theta}(\mu)$.

Theorems 1.2 and 1.3 similarly follow with Corollaries 6.1 and 6.2, respectively.

7. EXAMPLES

We briefly list a few examples to which our results may be applied. The main focus lies on Theorem 1.2 with nonlinear multiplicative noise term. (Recall that the hypotheses of Theorem 1.3 for the linear case are less restrictive. In particular, for all spectral dimensions one can consider sufficiently regular noises z which admit a function solution of the corresponding equation.)

1. Semigroups on $\mathbb{R}^n$. Suppose $X = \mathbb{R}^n$, $n \geq 3$, $\mathcal{X}$ is the Borel σ-field and $\mu(dx) = dx$ is the n-dimensional Lebesgue measure.

(i) Consider the Markovian semigroup $(T(t))_{t\geq 0}$ associated to the classical Dirichlet form

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^n} \nabla u(x) \nabla v(x) dx$$

with domain $H^1(\mathbb{R}^n)$ (in this case the classical Sobolev space $W_2^1(\mathbb{R}^n)$). cf. [17]. The corresponding generator is $\frac{1}{2}\Delta$. If $\alpha < \frac{1}{2} - \frac{n}{8}$ one can find some β and $q > 2$ such that for $z \in C^{1-\alpha}([0,t_0], H_q^{-\beta}(\mathbb{R}^n))$, problem (1) with $\theta = 1$ has a function solution in $W^\gamma([0,t_0], H_\infty^\delta(\mathbb{R}^n))$ for some γ and δ.

(ii) If we subject $\mathcal{E}$ resp. $(T(t))_{t\geq 0}$ to subordination with parameter $0 < \theta < 1$, we obtain corresponding non-local objects $\mathcal{E}^{(\theta)}$ resp. $(T^{(\theta)}(t))_{t\geq 0}$. In particular,

$$\mathcal{E}^{(\theta)}(u,v) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(x)-u(y))(v(x)-v(y))|x-y|^{-n-2\theta} dxdy$$

with domain $H^\theta(\mathbb{R}^n)$ and generator

$$-A^\theta u(x) = \lim_{\varepsilon\to 0} \frac{1}{2}\int_{|x-y|<\varepsilon} (u(y)-u(x))|x-y|^{-n-2\theta} dy \ .$$

The corresponding semigroup is ultracontractive with spectral dimension $\frac{n}{\theta}$. Function solutions may occur if $\alpha < \frac{1}{2} - \frac{n}{8\theta}$, then in $W^\gamma([0,t_0], H_\infty^{\delta\theta}(\mathbb{R}^n))$. Note that in particular we need $\theta > \frac{n}{4}$.

(iii) For fixed $0 < \theta < 1$ superposition yields a Dirichlet form

$$\widetilde{\mathcal{E}} := \mathcal{E} + \mathcal{E}^{(\theta)} \ .$$

The corresponding generator is an integro-differential operator $-A - A^\theta$ that gives rise to a self-adjoint operator in $L_2(\mathbb{R}^n)$. It is known that $\mathcal{E}$ has domain $H^1(\mathbb{R}^n)$. The corresponding Markovian semigroup is ultracontractive with spectral dimension $\frac{n}{\theta}$, i.e. the jump part dominates at zero, see for instance [11]. We need again $\alpha < \frac{1}{2} - \frac{n}{8\theta}$ for solvability, but solutions can be in $W^\gamma([0,t_0], H_\infty^\delta(\mathbb{R}^n))$.

2. Neumann semigroups on metric measure spaces. Let (X,d,μ) be a metric measure space as in (MMS).

(i) Assume $(T(t))_{t\geq 0}$ is a semigroup associated to a local regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on X, generated by a (fractal) Laplacian. For p.c.f. self-similar (as Sierpinski gaskets or Koch curves) such Dirichlet forms $\mathcal{E}$ can be constructed as rescaled limits of increasing sequences of discrete Dirichlet forms on approximating graphs, [27], [36]. For classes of nested fractals they may be obtained as the fixed point of a renormalization map, see e.g. [2]. Further constructions are studied in [28], [38], [3], [19]. In these examples the corresponding heat kernels are known to exist and satisfy (HKE(β)) for some (in fact, any) $\beta > 0$. Given a noise $z \in C^{1-\alpha}([0,t_0], H_q^{-\beta}(\mu))$, we may have functions solutions for $\theta = 1$ if $d_S = \frac{2d_f}{w} < 4$, where $d_f = dim_H X$ is the Hausdorff-dimension of X and w the so-called walk dimension of the semigroup, cf. [2]. To possibly have function solutions, we need $\alpha < \frac{1}{2} - \frac{d_S}{8}$.

(ii) As before, one may construct non-local Dirichlet forms by subordination with parameter $0 < \theta < 1$, leading to similar results as for the real line. Note however that in general the form

$$\mathcal{E}^{(\theta)}(u,v) = \int_X \int_X (u(x)-u(y))(v(x)-v(y)) d(x,y)^{-d_f-2\theta} \mu(dx)\mu(dy) \ ,$$

$0 < \theta < 1$, belongs to a different semigroup, [30]. Nevertheless we need $\alpha < \frac{1}{2} - \frac{d_S}{8\theta}$ and in particular $d_S < 4\theta$ in either case.

(iii) Superposition yields

$$\widetilde{\mathcal{E}} := \mathcal{E} + \mathcal{E}^{(\theta)}$$

having spectral dimension $\frac{d_S}{\theta}$ as can be deduced from known heat kernel estimates, see e.g. [11]. We may obtain function solutions under similar conditions as before.

3. Dirichlet, Neumann and censored semigroups on bounded domains in $\mathbb{R}^n$. Further special cases arise from boundary vaue problems.

(i) For instance onsider the problem

$$\begin{cases} \frac{\partial u}{\partial t} = -(-\frac{1}{2}\Delta_D)^\theta u + F(u) + G(u)\frac{\partial z}{\partial t} & \text{on } D \text{ and for } t \in (0,t_0) \\ u(0,x) = f \ \ , \ t \in (0,t_0) \\ u(t,x) = 0 \ \ , \ x \in \partial D \end{cases} \tag{47}$$

Here $D \subset \mathbb{R}^n$, $n \leq 3$, is a smooth bounded domain, Δ_D denotes the Dirichlet Laplacian with its usual domain $H^2_{2,0}(D)$, cf. [22], and we consider the associated semigroup. For $\theta = 1$ Theorem (1.1) may be applied to obtain mild solutions to the Dirichlet problem (47). Function solutions are seen to be possible if $\alpha < \frac{1}{2} - \frac{n}{8}$. If we consider the fractional powers $(-\frac{1}{2}\Delta_D)^\theta$, $0 < \theta < 1$ and their (subordinated) semigroups, we may solve related abstract Cauchy problems under similar conditions as above. If we replace the Dirichlet boundary conditions by Neumann boundary conditions and consider the Neumann Laplacian Δ_N with domain $H^2_{\partial/\partial\nu}(D)$ in place of Δ_D, we get similar results for $0 < \theta \leq 1$. For these classes of semigroups we refer to [26].

(ii) On the other hand we may consider the fractional Laplacian $A := (-\frac{1}{2}\Delta)^\theta$ wtih $0 < \theta < 1$ fixed on $\mathbb{R}^n$, $n \leq 3$ and study the problem

$$\begin{cases} \frac{\partial u}{\partial t} = -A_D u + F(u) + G(u)\frac{\partial z}{\partial t} & \text{on } D \text{ and for } t \in (0, t_0) \\ u(0,x) = f \ , \ t \in (0, t_0) \\ u(t,x) = 0 \ , \ x \in D^c \ . \end{cases} \tag{48}$$

Here D is a bounded domain in $\mathbb{R}^n$, $n \leq 3$, not necessarily smooth, $-A_D$ denotes the generator of the Dirichlet form

$$\mathcal{C}(u,v) = \frac{c_\theta}{2}\int_D\int_D (u(x)-u(y))(v(x)-v(y))|x-y|^{-n-2\theta}dxdy + \int_D u(x)v(x)\kappa(x)dx$$

with domain $\mathcal{D}(\mathcal{C}) = \{u \in H^\theta(\mathbb{R}) : u = 0 \ \text{ q.e. on } D^c\}$ and

$$\kappa(x) = c_\theta \int_{D^c} \frac{dy}{|x-y|^{n+2\theta}} \ .$$

$c_\theta > 0$ is a suitable constant. This corresponds to the killed stable semigroup on $L_2(D)$ which is ultracontractive with spectral dimension $\frac{n}{\theta}$, see [23] or [8]. The condition for function solutions are as before.

(iii) Dropping the killing term, one may also consider the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = -A_C u + F(u) + G(u)\frac{\partial z}{\partial t} & \text{on } D \text{ and for } t \in (0, t_0) \\ u(0,x) = f \ , \ t \in (0, t_0) \\ u(t,x) = 0 \ , \ x \in D^c \ , \end{cases} \tag{49}$$

where A_C is the generator of the regular Dirichlet form on $L_2(D)$ given by

$$\widetilde{\mathcal{C}}(u,v) = \frac{c_\theta}{2}\int_D\int_D (u(x)-u(y))(v(x)-v(y))|x-y|^{-n-2\theta}dxdy$$

with domain $\mathring{H}^\theta(D)$. Usually the corresponding semigroup is referred to as the censored semigroup. See [8].

4. Discrete problems. Further interesting applications arise if X, for instance endowed with the counting measure μ, is the discrete state space of a continuous time Markov chain with transition semigroup $(T(t))_{t\geq 0}$ associated to a Dirichlet form of type

$$\mathcal{E}(u,v) = \sum_{y\neq x}(u(x)-u(y))(v(x)-v(y))c_{xy} \ ,$$

where $c_{xy} = c_{yx} \in [0, \infty)$ are prescribed numbers that may be interpreted as conductances between points x and y. For instance X may be the vertex set of a suitable graph. For simplicity we just point out two examples on $\mathbb{Z}^n$, $n \leq 3$: If the c_{xy} satisfy some non-degeneracy and local irreducibility conditions and obey a uniform second moment condition, the spectral dimension is known to be n, [4], Proposition 3.1. This example corresponds to the diffusion case, a special case is the simple random walk associated to the graph Laplacian, where $c_{xy} = 1$ if x and y are neighbours and zero otherwise. If the c_{xy} are comparable to $|x-y|^{-n-2\theta}$ then the Dirichlet form $\mathcal{E}$ is comparable to that of a stable process, see [5], the spectral dimension is $\frac{n}{\theta}$. In both cases the conditions for function solutions are as before.

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