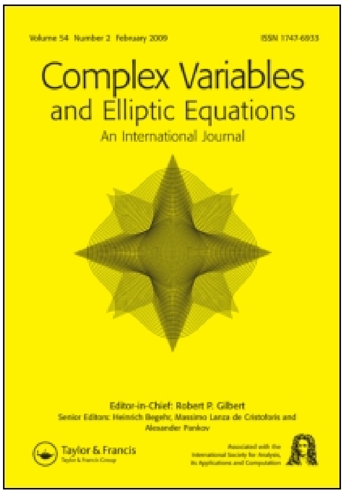

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Gradient-type noises I-partial and hybrid integrals

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Gradient-type noises I – partial and hybrid integrals

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The final objective of our study is to propose a finite-dimensional approach to systems of parabolic partial differential equations perturbed by low-order noises of Brownian or fractional Brownian type. The present article is the preparatory first part, where we introduce partial pathwise integrals over D and $(a,b)\times D$, where D is a smooth bounded domain in $\mathbb{R}^n$. Corresponding stochastic versions appear as limit cases.

Keywords: stochastic partial differential equations; fractional Brownian sheet; fractional calculus; function spaces; Stieltjes integrals

AMS Subject Classifications: 26A33; 26A42; 60H05; 60H15

1. Introduction

The present article is the first part of a paper concerned with a pathwise approach to stochastic partial differential equations. In a finite-dimensional setting, systems of partial differential equations will be considered under certain random noises. An example is the parabolic problem with multiplicative noise given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u\cdot\dfrac{\partial^2 Z}{\partial t\partial x}, \\ u(0,x)=f(x),\ \ x\in(a,b), \\ u(t,a)=u(t,b)=0,\ \ t\in(0,T), \end{cases} \tag{1}$$

where $T>0$ and $(a,b)\subset\mathbb{R}$ is a finite interval. Here f: $(a,b)\to\mathbb{R}$ is a suitable initial condition and $\frac{\partial^2 Z}{\partial t\partial x}$ is interpreted as a space-time noise arising from some given random field $Z=(Z(t,x))_{(t,x)\in\mathbb{R}^2}$. We will say (1) possesses the (*mild*) *solution* $u:[0,T]\times(a,b)\to\mathbb{R}$ if

$$u(t)=P_t f + I_t\left(u,\frac{\partial^2 Z}{\partial t\partial x}\right) \tag{2}$$

is well defined as a member of a suitable function space. $(P_t)_{t\geq 0}$ denotes the transition semigroup of Brownian motion killed upon leaving (a,b) and $I_t(u,\frac{\partial^2 Z}{\partial t\partial x})$ is some integral operator which to give a meaning to is the central matter.

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Apart from restrictions on α and β in order to solve (1), it will for instance be possible to consider $Z := B^{\alpha,\beta}$, where $B^{\alpha,\beta} = (B^{\alpha,\beta}(t,x))_{(t,x)\in\mathbb{R}^2}$ is the *anisotropic fractional Brownian sheet* in the sense of [1,2], see also [3]. The special case with $\alpha = \beta = 1/2$ yields the *Brownian sheet* $W = (W(t,x))_{(t,x)\in\mathbb{R}^2}$, and our integral construction will be such that if W is chosen as integrator, we (essentially) arrive at the stochastic Itô integral used in the spatially one-dimensional examples in [4].

We will also consider space dimensions $n > 1$. Recall that in the classical approach of [4], the $(n+1)$-dimensional Brownian sheet $W = (W(t,x))_{(t,x)\in\mathbb{R}^{n+1}}$ as integrator describes space-time white noise, formally given by the full mixed derivative $(\partial^{n+1}W)/(\partial t\partial x_1 \ldots \partial x_n)$ of W. It is known that for space dimension $n > 1$, the solutions to higher dimensional analogues of (1) involving space-time white noises cannot be scalar-valued processes, cf. e.g. [4] or [5].

We propose to study lower order noises formally given by mixed partial derivatives $\frac{\partial}{\partial t}\nabla Z$ where $\nabla g = (\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n})$ denotes the (*spatial*) *gradient* of an $\mathbb{R}$- or $\mathbb{R}^k$-valued function g on $[0,T] \times D$, interpreted in distributional sense. Z can be replaced by the paths of a suitable random field, in particular, by some (hybrid) fractional Brownian sheets or fields, see [2,3,6]. Some interesting motivations will be pointed out in part II of this study, [7], where we will actually solve some problems including (1).

The purpose of the present part I is to survey some background for the definition of the pathwise integral operator $I_t(u, \frac{\partial^2 Z}{\partial t\partial x})$ which, together with higher dimensional analogues, will be introduced in part II. We describe basic types of integrals over D or $(a,b) \times D$, where D is a bounded C^∞-domain in $\mathbb{R}^n$. We employ some two-parameter Stieltjes-type integration. For the one-parameter case, see [8–16] and the references therein. Since we aim at a construction well suited to PDE theory, we follow [12,14–16] and use a combination of fractional calculus and Fourier analysis. We stress that we stay within the 'case of simple duality' which does not require the application of iterated integrals or rough path methods.

Forward integrals of formal type

$$\int\int_{(a,b)\times D} \left\langle f, (\frac{\partial}{\partial t}\nabla)^+ g \right\rangle \mathrm{d}(t,x) \tag{3}$$

for $\mathbb{R}^n$- and $\mathbb{R}$-valued non-random functions f and g, respectively, on $(a,b) \times D$ are introduced. Paths of certain (hybrid) fractional Brownian sheets $B^{\alpha,\beta}$ can be considered in place of f and g. In the case where the integrator g is a (hybrid) Brownian sheet W, averaging and convergence in square mean yield a related stochastic integral

$$(A)\int\int_{(a,b)\times D} \left\langle H, (\frac{\partial}{\partial t}\nabla)^+_{\mathbb{P}} W \right\rangle \mathrm{d}(t,x)$$

for bounded and predictable integrands H, cf. [12,15]. When $D = (a,b)$ is an interval, this stochastic integral then coincides with the two-parameter Itô integral (of the first kind) as considered e.g. in [4,17], or [18].

In Section 2, we give a survey on fractional calculus for functions with values in separable complex Banach spaces. This allows to define Stieltjes-type integrals for such functions in the same way as it was done for scalar-valued in [14], outlined in Section 3. In Section 4, we consider forward integrals for functions on smooth domains and give an existence statement. In Section 5, we 'clip' the approaches and obtain integrals for functions on $(a,b) \times D$. In Section 6, we consider fractional Brownian sheets and fields and integrate one against another in the pathwise sense. In Section 7, where a Brownian sheet serves as integrator, a stochastic integral is defined as an average limit taken in the mean square. It is shown to coincide with the two-parameter Itô integral of the first kind.

The proofs are shifted to Appendix 1 and sometimes given for special cases only, their generalizations being a matter of notation. In Appendix 2, we briefly describe Stieltjes integrals in Banach spaces in the sense of [19], as they are related to the integrals from Section 3 in the expected manner. Positive constants whose values are not of importance are denoted by c.

2. Notions from fractional calculus

Let X be a separable complex Banach space with norm $\|\cdot\|_X$ and let I be an interval or $\mathbb{R}$. In later applications it will be possible to restrict the attention to real subspaces of X.

For $1 \leq p < \infty$ let $L_p(I, X)$ denote the space of (equivalence classes of) measurable functions $f\colon I \to X$ such that

$$\|f|L_p(I,X)\| = \left(\int_I \|f(t)\|_X^p \mathrm{d}t\right)^{1/p} < \infty ,$$

the integrals taken in the sense of Bochner, see e.g. [20]. We write $L_1(I)$ in case $X = \mathbb{C}$. All proofs of the facts we quote in the following carry over from the scalar-valued case, we refer the reader to [21].

2.1. *Fractional integrals and derivatives on an interval*

Let $I = (a, b)$ be a bounded interval. Given $\alpha > 0$ and a function $\varphi \in L_1((a, b), X)$, consider the (forward and backward) *Riemann–Liouville fractional integrals of order* α by

$$I_{a+}^{\alpha}\varphi(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau$$

and

$$I_{b-}^{\alpha}\varphi(t) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{\varphi(\tau)}{(\tau-t)^{1-\alpha}} \mathrm{d}\tau .$$

Here for $\alpha > 0$ the powers are understood as usual in the sense of choosing the main branch of the analytic function z^{α}, $z \in \mathbb{C}$, with the cut along the positive half-axis. In particular, $(-1)^{\alpha} = e^{i\alpha\pi}$.

One has $I_{a+}^{\alpha} I_{a+}^{\beta} = I_{a+}^{\alpha+\beta}$ if α, $\beta \geq 0$ and $\lim_{\varepsilon\to 0} I_{a+}^{\varepsilon}\varphi = \varphi$ in $L_p((a,b), X)$, provided $\varphi \in L_p((a,b), X)$. The same is true for I_{b-}^{α}. Let $I_{a+}^{\alpha}(L_p((a,b),X))$ denote the space of functions $f = I_{a+}^{\alpha}\varphi$ with $\varphi \in L_p((a,b), X)$, similarly $I_{b-}^{\alpha}(L_p((a,b),X))$. For $0 < \alpha < 1$ and functions $f \in I_{a+}^{\alpha}(L_p((a,b),X))$, respectively $f \in I_{b-}^{\alpha}(L_p((a,b),X))$, consider the (forward, respectively backward) *Weyl–Marchaud fractional derivatives of order* α,

$$D_{a+}^{\alpha} f(t) := \frac{\mathbf{1}_{(a,b)}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t) - f(\tau)}{(t-\tau)^{\alpha+1}} \mathrm{d}\tau \right)$$

and

$$D_{b-}^{\alpha} f(t) := \frac{(-1)^{\alpha} \mathbf{1}_{(a,b)}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t) - f(\tau)}{(\tau - t)^{\alpha+1}} \mathrm{d}\tau \right),$$

the convergence of the principal values of the hypersingular integrals being pointwise almost everywhere if $p=1$ and in $L_p((a,b), X)$ if $p \geq 1$. Under these assumptions $I_{a+}^{\alpha} D_{a+}^{\alpha} f = f$ in $L_p((a,b), X)$, while $D_{a+}^{\alpha} I_{a+}^{\alpha} \varphi = \varphi$ is true for any $\varphi \in L_1((a,b), X)$. This may be completed in the case $\alpha = 1$ by putting $D_{a+}^1 f = \mathrm{d}f/\mathrm{d}x$ and $D_{b-}^1 f = -\mathrm{d}f/\mathrm{d}x$ and in the case $\alpha = 0$ by the identity. The space $I_{a+}^{\alpha}(L_p((a,b), X))$ will be endowed with the norm

$$\| f | I_{a+}^{\alpha}(L_p((a,b), X)) \| := \| D_{a+}^{\alpha} f | L_p((a,b), X) \|, \tag{4}$$

similarly for $I_{b-}^{\alpha}(L_p((a,b), X))$.

2.2. *Integration-by-parts*

Now let E and F be two separable real Banach spaces normed by $\|\cdot\|_E$, and $\|\cdot\|_F$ respectively, and let $L = L(E, F)$ denote the Banach space of bounded linear operators from E into F endowed with the operator norm $\|\cdot\|_L$. Standard arguments prove the integration-by-parts rule for derivatives,

$$(-1)^{\alpha} \int_a^b U(t) D_{a+}^{\alpha} f(t) \mathrm{d}t = \int_a^b D_{b-}^{\alpha} U(t) f(t) \mathrm{d}t, \tag{5}$$

provided $0 \leq \alpha \leq 1, p, q \geq 1$, $1/p + 1/q < 1 + \alpha$, $f \in I_{a+}^{\alpha}(L_q((a,b), E))$ and $U \in I_{b-}^{\alpha}(L_p((a,b), L))$, the limiting cases for α justified by ordinary calculus.

3. Stieltjes-type integrals in Banach spaces

Again, let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two separable real Banach spaces and $L = L(E, F)$ the space of bounded linear operators normed by $\|\cdot\|_L$.

3.1. *Stieltjes-type integrals*

To build up a type of integral as needed in part II, we start with a Stieltjes integral as introduced [14], now for vector-valued functions. For f: $(a,b) \to E$ we suppose $f(a+) = \lim_{\delta \to 0} f(a+\delta)$ exists in the strong sense and put $f_{a+} = \mathbf{1}_{(a,b)}(t)(f(t) - f(a+))$. For $U: (a,b) \to L$ we assume that $U(a+)f = \lim_{\delta \to 0} U(a+\delta)f$ for any $f \in E$ exists as the limit in the strong sense in F, then consequently $U(a+) \in L$, too. Set $U_{a+}(t) = \mathbf{1}_{(a,b)}(t)(U(t) - U(a+))$. The meanings of f_{b-} and U_{b-} are similar.

Definition 3.1 Suppose $0 \leq \alpha \leq 1$, $p, q \geq 1$, $1/p + 1/q \leq 1$. Let f: $(a,b) \to E$ be such that $f_{a+} \in I_{a+}^{\alpha}(L_p((a,b), E))$ and U: $(a,b) \to L$ such that $U_{b-} \in I_{b-}^{1-\alpha}(L_q((a,b), L))$. Define the *forward integral* of the E-valued function f with respect to the operator-valued function U by

$$\int_a^b f(t)\mathrm{d}U(t) := (-1)^\alpha \int_a^b D_{b-}^{1-\alpha} U_{b-}(t) D_{a+}^{\alpha} f_{a+}(t)\mathrm{d}t \\ + U(b-)f(a+) - U(a+)f(a+) \qquad (6)$$

The integral is directed forward.

Remark 3.2

(i) As in [14], (5) can be used to show that the definition (6) is correct, i.e. does not depend on the particular choice of $0 \leq \alpha \leq 1$.

(ii) To justify the notation of the marginal limits in (6), apply the triangle inequality to $\| U(a+\varepsilon)f(a+\delta) - U(a+)f(a+)\|_F$ for $\varepsilon > 0$ and $\delta > 0$ and consider $\|(U(a+\varepsilon) - U(a+))f(a+)\|_F$ as well as $\|U(a+\varepsilon)(f(a+\delta) - f(a+))\|_F$. For the first term obviously the order of the limit processes is arbitrary, for the second term this follows from the existence of $U(a+) \in L$ in the strong sense and since $U(t) \in L,\ t \in (a,b)$.

(iii) In the case that $0 \leq \alpha < 1/p$, the entire right-hand side in (6) equals the integral with just f in place of f_{a+} (then without correction terms).

(iv) If real Banach spaces are considered, the integrals given by (6) are real-valued, due to the definition of the fractional derivatives.

The integrals defined this way extend Riemann–Stieltjes integrals in Banach spaces as studied in [19], see Appendix 2.

Remark 3.3 In part II $(U(t))_{t\geq 0}$ will be a suitable operator semigroup on E. Under hypotheses more familiar than those of Definition 3.1, a related type of integral will be defined, see Remark 5.5 in Section 5.

3.2. *A special case*

Let E' denote the dual space of E. For $g \in E'$ let $\langle f, g\rangle$ denote the *dual pairing* of $f \in E$ and $g \in E'$. Given a function $g:(a,b) \to E'$.

$$U(t) := \langle \cdot, g(t) \rangle, \quad t \in (a,b), \qquad (7)$$

defines a bounded operator-valued function $U:(a,b) \to E$. We assume there is some $g(a+) \in E'$ such that $\langle f, g(a+)\rangle = \lim_{\delta\to 0} \langle f, g(a+\delta)\rangle$ for any $f \in E$ and put $g_{a+}(t) = \mathbf{1}_{(a,b)}(t)(g(t) - g(a+))$. Specifying Definition 3.1 to this case with $g:(a,b) \to E'.\ g_{b-} \in I_{b-}^{1-\alpha}(L_q((a,b),E'))$ with α, p, q and f as before, the forward integral according to (6) equals

$$\int_a^b \langle f(t), \mathrm{d}g(t)\rangle := (-1)^\alpha \int_a^b \langle D_{a+}^{\alpha} f_{a+}(t), D_{b-}^{1-\alpha} g_{b-}(t)\rangle \mathrm{d}t \\ + \langle f(a+), g(b-)\rangle - \langle f(a+), g(a+)\rangle\ . \qquad (8)$$

For $E = E' = \mathbb{R}$ we arrive at the integral considered in [14] and [15].

3.3. *Average integrals*

Finally, we state two limit representations and define an average version.

LEMMA 3.4 *Let* $1/p+1/q\leq 1$.

(i) *Under the assumptions of Definition* 3.1 *we have*

$$\int_a^b f(t)\mathrm{d}U(t)=\lim_{\varepsilon\to 0}\int_a^b I_{a+}^{\varepsilon}f(t)\mathrm{d}U(t).$$

(ii) *Suppose* $0<\alpha<1$, $0<\varepsilon<\alpha$, $f\colon (a,b)\to E$ *is such that* $f\in I_{a+}^{\alpha-\varepsilon}(L_p((a,b),E))$, $\alpha p\neq 1$, *and* $U\colon (a,b)\to L$ *is such that* $U_{b-}\in I_{b-}^{1-\alpha}(L_q((a,b),L))$. *Then we have*

$$\int_a^b I_{a+}^{\varepsilon}f(t)\mathrm{d}U(t)=\frac{(1-\varepsilon)}{\Gamma(\varepsilon)}\int_0^{\infty}t^{\varepsilon-1}\int_a^b(U_{b-}(s+t)-U_{b-}(s))f(s)\,\mathrm{d}s\frac{\mathrm{d}t}{t},$$

the integral $\int_0^{\infty}$ *taken in the sense of principal values.*

The proof of the lemma resembles that of the scalar-valued case, we refer to [15], Lemmas 4.1 and 4.2.

In view of the asymptotics of the Gamma function ε may replace $1/\Gamma(\varepsilon)$ when considering the limit as $\varepsilon\to 0$. We set

$$(A)\int_a^b f(t)\mathrm{d}U(t):=\lim_{\varepsilon\to 0}\varepsilon\int_0^1 t^{\varepsilon-1}\int_a^b(U_{b-}(s+t)-U_{b-}(s))f(s)\mathrm{d}s\frac{\mathrm{d}t}{t}\tag{9}$$

for measurable functions $f\colon(a,b)\to E$ and $U\colon(a,b)\to L$ whenever the right-hand side is well defined and exists. This is an extension of Definition 3.1. The notation (A) means 'average'.

4. Gradient-type integrals via duality

This section introduces forward integrals of $\mathbb{R}^n$-valued fields f w.r.t. $\mathbb{R}$-valued fields g over smooth bounded domains $D\subset\mathbb{R}^n$. The special case $n=1$ yields the forward integral as familiar from [12].

First some preliminary facts about function spaces are recalled. Then the forward integral is defined and using simple Fourier multiplier arguments, conditions sufficient for its existence are proved. Finally, limit representations are shown, leading to an average version.

Note that below we use spaces of complex-valued functions and distributions, but *all results remain valid if the functions are taken to be real-valued.*

4.1. *Preliminaries*

$\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}^n$. The Fourier transform of $f\in\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$ is denoted by $f\mapsto f^{\wedge}$, its inverse by $f\mapsto f^{\vee}$. For $1<p<\infty$ and $\alpha\in\mathbb{R}$, the *Bessel potential spaces of order* α are defined by

$$H_p^{\alpha}(\mathbb{R}^n):=\left\{f\in\mathcal{S}'(\mathbb{R}^n):\left\|f|H_p^{\alpha}(\mathbb{R}^n)\right\|<\infty\right\},$$

$$\left\|f|H_p^{\alpha}(\mathbb{R}^n)\right\|:=\left\|\left((1+|\xi|^2)^{\alpha/2}f^{\wedge}\right)^{\vee}|L_p(\mathbb{R}^n)\right\|.\tag{10}$$

Here $|\cdot|$ is used for the Euclidean norm on $\mathbb{R}^n$. For $\sigma\in\mathbb{R}$, the linear operator I_σ,

$$I_\sigma f = \left((1+|\xi|^2)^{\sigma/2} f^\wedge\right)^\vee , \tag{11}$$

is an isomorphism of $H_p^\alpha(\mathbb{R}^n)$ onto $H_p^{\alpha-\sigma}(\mathbb{R}^n)$. For $\alpha>0$, $(1+|\xi|^2)^{-\alpha/2}$ is the Fourier image of the *Bessel kernel* $G_\alpha\in L_1(\mathbb{R}^n)$ and $I_{-\alpha}f$, $f\in L_p(\mathbb{R}^n)$, $1<p<\infty$, according to (11) is realized as the convolution $G^\alpha f := G_\alpha * f$. The dual space of $H_p^\alpha(\mathbb{R})$ is $H_{p'}^{-\alpha}(\mathbb{R})$, $1/p+1/p'=1$. For $u\in H_p^\alpha(\mathbb{R}^n)$ and $v\in H_{p'}^{-\alpha}(\mathbb{R}^n)$, the dual pairing of u and v is given by $\langle u,v\rangle$ and

$$|\langle u,v\rangle| \le c\left\| u|H_p^\alpha(\mathbb{R}^n)\right\| \left\| v|H_{p'}^{-\alpha}(\mathbb{R}^n)\right\| . \tag{12}$$

For our purposes it seems convenient to introduce also another type of space. Let $\{e_1,\dots,e_n\}$ denote the standard basis in $\mathbb{R}^n$. For $x=(x_1,\dots,x_n)$ and fixed $l=1,\dots,n$, write $x_1'=(x_1,\dots,x_{l-1},x_{l+1},\dots,x_n)$ for the $(n-1)$-vector obtained from x by disposing the coordinate x_l, and identify x with (x_l',x_l), $x=(x_l',x_l)$. For $l=1,\dots,n$ fixed $f\mapsto f^{\wedge_l}$ denotes the partial Fourier transform of f with respect to x_l, i.e.

$$f^{\wedge_l}(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}_l} e^{ix_l\xi_l} f(\xi_l', x_l)\mathrm{d}x_l\ , \quad f\in\mathcal{S}(\mathbb{R}^n), \quad \xi\in\mathbb{R},$$

where $\mathbb{R}_l := \text{span}\ \{e_l\}$. $f\mapsto f^{\vee_l}$ denotes its inverse. It follows that for a C^∞-function $m(\xi_l)$ depending only on ξ_l and being of polynomial growth, we always have $(m(\xi_l)f^{\wedge_l})^{\vee_l} = (m(\xi_l)f^\wedge)^\vee$, where $f\in\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$, see [22]. By $H_{p,l}^\alpha(\mathbb{R}^n)$, $1<p<\infty$, $\alpha\in\mathbb{R}$, we denote the space

$$H_{p,l}^\alpha(\mathbb{R}^n) := \left\{ f\in\mathcal{S}'(\mathbb{R}^n)\colon \left\| f|H_{p,l}^\alpha(\mathbb{R}^n)\right\| <\infty\right\},$$

where

$$\left\| f|H_{p,l}^\alpha(\mathbb{R}^n)\right\| := \left\| \left((1+\xi_l^2)^{\alpha/2} f^\wedge\right)^\vee | L_p(\mathbb{R}^n)\right\|. \tag{13}$$

For $\alpha\ge 0$, $H_p^\alpha(\mathbb{R}^n)$ is continuously embeded in $H_{p,l}^\alpha(\mathbb{R}^n)$, for $\alpha<0$ we have the converse embedding. The space $\mathcal{S}(\mathbb{R})$ is dense in both spaces. Given $\alpha_l\in\mathbb{R}$, $l=1,\dots,n$, put $\overline{\alpha}=(\alpha_1,\dots,\alpha_n)$ and

$$H_p^{\overline{\alpha}}(\mathbb{R}^n) := \bigcap_{l=1}^n H_{p,l}^{\alpha_j}(\mathbb{R}^n),$$

which are Banach spaces if normed by $\| f|H_p^{\overline{\alpha}}(\mathbb{R}^n)\| := \sum_{l=1}^n \| f|H_{p,l}^{\alpha_l}(\mathbb{R}^n)\|$.

4.2. *Forward integrals*

Let $D\subset\mathbb{R}^n$ be a bounded domain. Let $f=(f_1,\dots,f_n)$ be an $\mathbb{R}^n$-valued vector field on $\mathbb{R}^n$ and g an $\mathbb{R}$-valued function on $\mathbb{R}^n$. For fixed $l=1,\dots,n$ denote the '*forward differences*' of g in direction e_l by

$$\partial_{l,r}^+ g(x) := \frac{1}{r}(g(x+re_l)-g(x))\ , \quad r>0. \tag{14}$$

Define the '*forward pre-gradient*' $\nabla_r^+ g$, $r > 0$, of the function g by

$$\nabla_r^+ g(x) := \left(\partial_{1,r}^+ g(x), \ldots, \partial_{n,r}^+ g(x)\right), \quad r > 0. \tag{15}$$

Writing $\langle \cdot, \cdot \rangle$ for the standard scalar product, we have $\langle f(x), \nabla_r^+ g(x) \rangle = \sum_{l=1}^n f_l(x), \partial_{l,r}^+ g(x)$ for $f = (f_1, \ldots, f_n)$.

Definition 4.1 Given an $\mathbb{R}^n$-valued field $f = (f_1, \ldots, f_n)$ on $\mathbb{R}^n$ and an $\mathbb{R}$-valued function g on $\mathbb{R}^n$, the (*partial*) *forward integral of f w.r.t. to g on D* is defined as the limit

$$\int_D \langle f(x), \nabla^+ g(x) \rangle \, dx := \lim_{r \to 0} \int_D \langle f(x), \nabla_r^+ g(x) \rangle \, dx, \tag{16}$$

whenever it exists. We also use $\int_D \langle f, \nabla^+ g \rangle \, dx$ to denote this integral.

Examples 4.2 Suppose $n = 1$, $D = (a, b)$, f and g both are defined on $\mathbb{R}$, g continuous at b. Let

$$\int_a^b f d^- g := \lim_{r \to 0} \int_a^b f(x) \frac{g_b(x + r) - g_b(x)}{r} dx$$

with $g_b := \mathbf{1}_{(a,b)}(x)\ (g(x) - g(b))$ denote the forward integral of f w.r.t. g as introduced in [12], whenever the limit exists. Then

$$\int_a^b f d^- g = \int_{(a,b)} \langle f, \nabla^+ g \rangle \, dx.$$

We prefer $+$, indicating the right-sided derivative instead of the traditional '$-$' referring to the integration process.

4.3. *Existence conditions*

As before, let $f = (f_1, \ldots, f_n)$ be an $\mathbb{R}^n$-valued field on $\mathbb{R}^n$ and g a $\mathbb{R}$-valued function on $\mathbb{R}^n$. Usually f and g will not be differentiable, but members of some function spaces. Denoting by $\mathbf{1}_D$ the indicator function of D, we first record the following:

LEMMA 4.3 *Let $D \subset \mathbb{R}^n$ be a bounded C^∞-domain and $h \in H_p^\alpha(\mathbb{R}^n)$ with $1 < p < \infty$, $0 < \alpha < 1/p$. Then*

$$\left\| \mathbf{1}_D h | H_p^\alpha(\mathbb{R}^n) \right\| \leq c \left\| h | H_p^\alpha(\mathbb{R}^n) \right\|$$

with a constant $c > 0$ independent of h.

This is proved in [23]. Here it is not necessary that D is C^∞, C^1 would be sufficient. Consider the *gradient* ∇g of g,

$$\nabla g := \left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right),$$

the partials taken in distributional sense. If g is such that for some $\alpha_l > 0$, $\frac{\partial g}{\partial x_l} \in H_{p',l}^{-\alpha_l}(\mathbb{R}^n) \subset H_{p'}^{-\alpha_l}(\mathbb{R}^n)$, and if f is such that $f_l \in H_p^{\alpha_l}(\mathbb{R}^n)$ with $0 < \alpha_l < 1/p$, $l = 1, \ldots, n$, then $\langle \mathbf{1}_D f_l, \frac{\partial g}{\partial x_l} \rangle$ may be seen as dual pairing according to (12). We write

$$\left\langle \mathbf{1}_D f, \nabla g \right\rangle := \sum_{l=1}^{n} \left\langle \mathbf{1}_D f_l, \frac{\partial g}{\partial x_l} \right\rangle. \tag{17}$$

By $\overline{1}$ we denote the n-vector $(1, \ldots, 1)$. In this situation we can slightly refine (12) to obtain:

PROPOSITION 4.4 *Let* $1 < p < \infty$, $1/p + 1/p' = 1$ *and let* $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *fulfil* $0 < \alpha_l \leq 1/p$, $l = 1, \ldots, n$. *Suppose* $f = (f_1, \ldots, f_n)$ *is such that* $f_l \in H_p^{\alpha_l}(\mathbb{R}^n)$ *and* $g \in H_{p'}^{\overline{1} - \overline{\alpha}}(\mathbb{R}^n)$. *Then the forward integral* (16) *exists and with notation* (17),

$$\int_D \left\langle f, \nabla^+ g \right\rangle \mathrm{d}x = \left\langle \mathbf{1}_D f, \nabla g \right\rangle.$$

Moreover, the estimate

$$\left| \int_D \left\langle f, \nabla^+ g \right\rangle \mathrm{d}x \right| \leq c \sum_{l=1}^{n} \left\| f_l | H_p^{\alpha_l}(\mathbb{R}^n) \right\| \left\| g | H_{p',l}^{1-\alpha_l}(\mathbb{R}^n) \right\|$$

holds.

The proof can be found in Appendix 1.

Remark 4.5 As in Definition (9), we may also consider the average limit corresponding to (16): Given two functions $f: \mathbb{R}^n \to \mathbb{R}^n$ and $g: \mathbb{R}^n \to \mathbb{R}$, set

$$(A) \int_D \left\langle f, \nabla^+ g \right\rangle \mathrm{d}x := \lim_{\varepsilon \to 0} \varepsilon \int_0^1 r^{\varepsilon - 1} \int_D \left\langle f(x), \nabla_r^+ g(x) \right\rangle \mathrm{d}x \, \mathrm{d}r, \tag{18}$$

whenever the right-hand side exists. (A) stands for 'average'. As the existence of (16) implies that of (18), the latter extends the first. For $n = 1$, $D = (a, b)$ and with f, g according to Examples 4.2, (18) reads

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^1 r^{\varepsilon - 1} \int_a^b f(x) \frac{g(x + r) - g(x)}{r} \mathrm{d}x \, \mathrm{d}r.$$

Recall the definition of the average integral via fractional calculus, (9). Specifying the case addressed in (7) and (8) further to such f, $g: \mathbb{R} \to \mathbb{R}$, these special cases of (18) and (9) yield the same.

5. Hybrid integrals

The aim of this section is to define a Stieltjes-type integral for functions of variables $(t, x) \in (a, b) \times D$, where (a, b) is a finite interval and D a bounded C^∞-domain in $\mathbb{R}^n$. We 'mix' the above constructions and use the approach via fractional calculus as in Section 3 taken with respect to the variable $t \in (a, b)$ and the approach via forward differences and duality as in Section 4 for the variable $x \in D$. Though seeming peculiar at first sight, this construction suits later studies of partial differential equations. There t will denote the time and x the space parameter.

In the special case where D itself is an interval, Stieltjes-type integrals can also be constructed using two-parameter fractional calculus. This can be carried through along the lines of [14], we refer to [21,24,25] and to [26] for an application to PDEs.

Below the hybrid integral is defined, and limit statements are deduced, needed for a representation of Itô-type integrals in Section 7. Functions depending on t and x will repeatedly be seen as vector-valued functions of t. If not mentioned otherwise, fractional calculus always applies to the variable $t \in (a,b)$. For the hybrid function spaces involved, we use the shortcut notation

$$\mathcal{H}^{\alpha,\beta}_{a+,p} := I^{\alpha}_{a+}(L_p((a,b), H^{\beta}_{p}(\mathbb{R}^n)) \ , \tag{19}$$

and

$$\mathcal{H}^{1-\alpha,\overline{1}-\overline{\beta}}_{b-,p'} := I^{\alpha}_{b-}(L_{p'}((a,b), H^{\overline{1}-\overline{\beta}}_{p'}(\mathbb{R}^n)) \ . \tag{20}$$

Assumptions Let $f = (f_1, \ldots, f_n)$: $(a,b) \times \mathbb{R}^n \to \mathbb{R}^n$ and g: $(a,b) \times \mathbb{R}^n \to \mathbb{R}$ be given. D is a bounded C^{∞}-domain in $\mathbb{R}^n$. We formulate conditions under which we introduce the hybrid integral. Assume that with some $1 < p < \infty$, $1/p + 1/p' = 1$ and $0 \le \alpha$, $\beta_l \le 1$, $\beta_l < 1/p$, $l = 1, \ldots, n$ the following holds:

(I) f_l and g possess the strong limits $f_l(a+)$, $g(a+)$ and $g(b-)$ in $H^{\beta_l}_{p}(\mathbb{R}^n)$ and $H^{\overline{1}-\overline{\beta}}_{p'}(\mathbb{R}^n)$, respectively.

(II) $f_{l,a+} \in \mathcal{H}^{\alpha,\beta_l}_{a+,p}$ and $g_{b-} \in \mathcal{H}^{1-\alpha,\overline{1}-\overline{\beta}}_{b-,p'}$, where $f_{l,a+}(x) := \mathbf{1}_{(a,b)}(x)(f_l(x) - f_l(a+))$ is as in Section 3, and g_{b-} is defined similarly.

Certain *Hölder properties imply these assumptions at once*, they are known to hold a.s. for samples of some random fields, see the next section. For $(t,x) \in (a,b) \times \mathbb{R}^n$, $u \in \mathbb{R}$ small enough in modulus and $r \in \mathbb{R}^n$, let

$$\Delta_{u,r}\varphi(t,x) := \varphi(t+u, x+r) - \varphi(t+u, x) - \varphi(t, x+r) + \varphi(t,x) \tag{21}$$

denote the 'rectangular' increments of a function φ on $(a,b) \times \mathbb{R}^n$. $|\cdot|_n$ denotes the Euclidean norm in $\mathbb{R}^n$, n is suppressed from notation if $n = 1$.

Lemma 5.1 *Let $U \subset \mathbb{R}$ be an open neighbourhood of $[a,b]$, $f: U \times \mathbb{R}^n \to \mathbb{R}^n$ and g: $U \times \mathbb{R}^n \to \mathbb{R}^n$, and assume there is a compact set $K \subset \mathbb{R}^n$ such that for any $t \in U$, the supports of f and g are contained in K. Let $0 < \alpha'' < \alpha < \alpha' < 1$ and $0 < \beta_l'' < \beta_l < \beta_l' < 1$, $l = 1, \ldots, n$. Suppose that both f and g fulfil the* multiple Hölder conditions

$$\left|\Delta_{u,r} f_l(t,x)\right| \le c|u|^{\alpha'}|r|_n^{\beta_l'} \ , \tag{22}$$

and

$$\left|\Delta_{u,se_l} g(t,x)\right| \le c|u|^{1-\alpha''}|s|^{1-\beta_l''} \ , \tag{23}$$

as well as the simple Hölder conditions

$$\left|f_l(t, x+r) - f_l(t,x)\right| \le c|r|_n^{\beta_l'}, \quad \left|f_l(t+u, x) - f_l(t,x)\right| \le c|u|^{\alpha'}, \tag{24}$$

and

$$\left|g(t, x + se_l) - g(t, x)\right| \leq c|s|^{\beta_l''}, \quad \left|g(t + u, x) - g(t, x)\right| \leq c|u|^{\alpha''}, \tag{25}$$

for all $(t, x) \in [a, b] \times \mathbb{R}^n$, *small* $s, u \in \mathbb{R}$, $r \in \mathbb{R}^n$ *and with a universal positive constant c. Then assumptions* (I) *and* (II) *are fulfilled.*

The proof is sketched in Appendix 1.

Definition 5.2 Suppose the functions f and g are as specified above and fulfil (I) and (II). We define the *hybrid* (*forward*) *integral* of f w.r.t. g over $(a, b) \times D$ by

$$\int\int_{(a,b)\times D} \left\langle f, \left(\frac{\partial}{\partial t}\nabla\right)^+ g\right\rangle d(t, x) := (-1)^\alpha \int_a^b \int_D \langle D^\alpha_{a+} f_{a+}(t), \nabla^+ D^{1-\alpha}_{b-} g_{b-}(t)\rangle \mathrm{d}x\, \mathrm{d}t + \int_D \langle f(a+), \nabla^+ g(b-)\rangle \mathrm{d}x - \int_D \langle f(a+), \nabla^+ g(a+)\rangle \mathrm{d}x, \tag{26}$$

the integrals over D with meaning as in (16). Here ∇^+ refers to x only.

Remark 5.3 The definition is correct, i.e. the value of the integral is independent of the values α and β_l. For β_l this follows from the proof of Proposition 4.4, for α it is similar to the one-parameter case [14].

PROPOSITION 5.4 *Under the assumptions of Definition* 5.2, *the integral in* (26) *exists. The spatial forward limit may then be replaced by the corresponding distributional derivative.*

In view of the norms (4) and (10), this is an immediate consequence of Proposition 4.4 together with Hölder's inequality.

Remark 5.5 Recall (2). Our definition of the integral operator $I_t(u, \frac{\partial^2 Z}{\partial t \partial x})$ in part II, [7], will be such that in terms of the transition densities $p(t, x, y)$ of the heat semigroup $(P(t))_{t\geq 0}$, it could at least formally be written in the above notation as

$$\int\int_{(0,t)\times(a,b)} \left\langle \psi, \left(\frac{\partial^2}{\partial t \partial y}\right)^+ Z\right\rangle d(s, y),$$

where $\psi(s) = p(t - s, x, \cdot)u(s, \cdot)$. Under suitable assumptions, this will be interpreted as

$$\int_0^t \int_{(a,b)} D^\alpha_{0+}\psi(s) \frac{\partial}{\partial y} D^{1-\alpha}_{t-} Z_t(s) \mathrm{d}y\, \mathrm{d}s$$

and can be evaluated by means of semigroup theory.

5.1. *Average integrals*

We wish to obtain an average version of (26), a special case of which will be used to represent the two parameter Itô integral in Section 7. The average integral is obtained by means of averaging with respect to the variable t.

LEMMA 5.6 *Under the assumptions of Definition* 5.2, *we have*

$$\int_a^b \int_D \langle D^{\alpha}_{a+} f(t), \nabla^+ D^{1-\alpha}_{b-} g_{b-}(t)\rangle \mathrm{d}x\,\mathrm{d}t = \lim_{\varepsilon\to 0} \int_a^b \int_D \langle D^{\alpha-\varepsilon}_{a+} f(t), \nabla^+ D^{1-\alpha}_{b-} g_{b-}(t)\rangle \mathrm{d}x\,\mathrm{d}t.$$

This follows straightforward using Propositions 4.4 and 5.4 together with the triangle inequality and the continuity properties of fractional integrals. We now use the additional assumption that

(III) For a.e. $x \in D$, $g(b-, x)$ exists and equals $g(b-)(x)$.

Under the hypotheses of Lemma 5.1, (III) is guaranteed.

LEMMA 5.7 *Let* $\varepsilon,\ r > 0,\ 0 < \alpha < 1$ *and* $0 < \beta < 1/p$. *Suppose* (I) *and* (III) *are valid,* $f_l \in \mathcal{H}^{\alpha-\varepsilon,\beta_l}_{a+,p}$, $l = 1, \ldots, n$, *and* $g_{b-} \in \mathcal{H}^{1-\alpha,1-\beta}_{b-,p'}$. *Then*

$$\frac{1-\varepsilon}{\Gamma(\varepsilon)} \int_0^\infty u^{\varepsilon-1} \int_a^b \int_D \left\langle f(t), \nabla^+_r \frac{g_{b-}(t+u) - g_{b-}(t)}{u} \right\rangle \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}u$$
$$= (-1)^\alpha \int_a^b \int_D \langle D^{\alpha-\varepsilon}_{a+} f(t), \nabla^+_r D^{1-\alpha}_{b-} g_{b-}(t)\rangle \mathrm{d}x\,\mathrm{d}t. \tag{27}$$

For the sake of legibility we formulate the proof only for the case $n = 1$ and $D = (a'.\,b')$, it can be found in Appendix 1. The case of general n follows by simple modifications.

Similar to the former sections we now define an average integral. For a function h: $(0, 1) \times (0, 1) \to \mathbb{R}$ we write $\lim_{|(\varepsilon,r)|\to 0} h(\varepsilon(t), r(t))$ if this limit exists and is independent of the particular path on which (ε, r) tends to the origin. Here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$. Set

$$(A) \int_{(a,b)\times D} \left\langle f, \left(\frac{\partial}{\partial t} \nabla\right)^+ g \right\rangle \mathrm{d}(t, x)$$
$$:= \lim_{|(\varepsilon,r)|\to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_a^b \int_D \left\langle f(t), \nabla^+_r \frac{g_{b-}(t+u) - g_{b-}(t)}{u} \right\rangle \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}u. \tag{28}$$

In view of Lemma 5.6 and 5.7, (28) is an extension of (26).

6. Pathwise integration of fractional Brownian sheets

In the course of this section, we survey some random fields and their path regularity and integrate them against each other in purely pathwise sense. Let $(\Omega, F, \mathbb{P})$ be a probability space.

6.1. *Fractional Brownian fields*

An $\mathbb{R}$-valued *fractional Brownian field* $B^\alpha = \{B^\alpha(x)\colon x \in \mathbb{R}^n\}$ of order $0 < \alpha < 1$ on $\mathbb{R}^n$ is a random field B^α: $\Omega \times \mathbb{R}^n \to \mathbb{R}$ such that B^α is Gaussian with mean zero, and the covariance function is given by

$$\mathbb{E}\{B^\alpha(x) B^\alpha(y)\} = \frac{1}{2}\left(|x|_n^{2\alpha} + |y|_n^{2\alpha} - |y - x|_n^{2\alpha}\right), \tag{29}$$

$x,\ y\in\mathbb{R}^n$. Recall that $|\cdot|_n$ denotes the Euclidean norm on $\mathbb{R}^n$. (29) in particular implies that a.s. $B^{\alpha}(0)=0$. We refer to [6]. For $\alpha=1/2$ we obtain Lévy's n-parameter Brownian motion, for $n=1$ the fractional Brownian motion, see [27,28]. From the covariance structure (29) one easily deduces that a.s. the paths of a suitable modification of B^{α} are α'-Hölder continuous on any compact set $K\subset\mathbb{R}^n$ and for any $0<\alpha'<\alpha$.

6.2. *Anisotropic fractional Brownian sheets*

An $\mathbb{R}$-valued *anisotropic fractional Brownian sheet* $B^{\overline{\alpha}}=\{B^{\overline{\alpha}}(t;x)\colon (t;x)\in\mathbb{R}^n\}$ on $\mathbb{R}^n$ of orders $\overline{\alpha}=(\alpha_1,\dots,\alpha_n)$, $0<\alpha_l<1$, $l=1,\dots,n$, is a random field $B^{\overline{\alpha}}\colon\Omega\times\mathbb{R}^n\to\mathbb{R}$ such that $B^{\overline{\alpha}}$ is Gaussian with mean zero, and the covariance function is given by

$$\mathbb{E}\{B^{\overline{\alpha}}(t,x)B^{\overline{\alpha}}(s,y)\}=\prod_{l=1}^{n}\frac{1}{2}\left(|x_l|^{2\alpha_l}+|y_l|^{2\alpha_l}-|y_l-x_l|^{2\alpha_l}\right),$$

$s,\ t\in\mathbb{R},\ x,\ y\in\mathbb{R}^n$. In particular $B^{\overline{\alpha}}(0)=0$ a.s. We refer to [1–3]. The case $\alpha_l=1/2$, $l=1,\dots,n$, yields the Brownian sheet on $\mathbb{R}^n$, see e.g. [4].

6.3. *(Hybrid) fractional Brownian sheets*

We mix the constructions and obtain Gaussian fields that will serve as standard examples in part II. An $\mathbb{R}$-valued (*hybrid*) *fractional Brownian sheet* $B^{\alpha,\beta}=\{B^{\alpha,\beta}(t,x)\colon (t,x)\in\mathbb{R}^{n+1}\}$ of orders $0<\alpha,\ \beta<1$ on $\mathbb{R}^{n+1}$ is a random field $B^{\alpha,\beta}\colon\Omega\times\mathbb{R}^{n+1}\to\mathbb{R}$ such that $B^{\alpha,\beta}$ is Gaussian with mean zero, and the covariance function is given by

$$\mathbb{E}\{B^{\alpha,\beta}(t,x)B^{\alpha,\beta}(s,y)\}=\frac{1}{2}\left(|t|^{2\alpha}+|s|^{2\alpha}-|t-s|^{2\alpha}\right)\frac{1}{2}\left(|x|_n^{2\beta}+|y|_n^{2\beta}-|y-x|_n^{2\beta}\right),\tag{30}$$

$s,\ t\in\mathbb{R},\ x,\ y\in\mathbb{R}^n$. Here $|\cdot|_n$ denotes the Euclidean norm on $\mathbb{R}^n$ and $|\cdot|$ the absolute value on $\mathbb{R}$. Again $B^{\alpha,\beta}(0)=0$ a.s. For the case $n=1$ we obtain an anisotropic sheet on $\mathbb{R}^2$.

In [1] it was shown that anisotropic sheets $B^{\alpha,\beta}$ on $\mathbb{R}^2$ have *stationary rectangular increments*, i.e. the law of the random variables

$$\Delta_{u,r}B^{\alpha,\beta}(t,x):=B^{\alpha,\beta}(t+u,x+r)-B^{\alpha,\beta}(t+u,x)-B^{\alpha,\beta}(t,x+r)+B^{\alpha,\beta}(u,x),$$

$u\in\mathbb{R},\ r\in\mathbb{R}$, does not depend on $(t,x)\in\mathbb{R}^{n+1}$. Their arguments easily carry over to anisotropic sheets $B^{\overline{\alpha}}$ on $\mathbb{R}^n$ or $B^{\alpha,\overline{\beta}}$ on $\mathbb{R}^{n+1}$, as well as to hybrid sheets $B^{\alpha,\beta}$ on $\mathbb{R}^{n+1}$. Using well-known moment properties of Gaussian random variables, and following the lines of [1] and [9] we deduce:

LEMMA 6.1 *Let $K\subset\mathbb{R}^n$ be compact and $[a,b]\subset\mathbb{R}$ as before.*

(1) *The hybrid sheet $B^{\alpha,\beta}$ on $\mathbb{R}^{n+1}$, $0<\alpha,\ \beta<1$, possesses a modification, again denoted by $B^{\alpha,\beta}$, whose paths on $[a,b]\times K$ a.s. fulfil the Hölder conditions* (22) *and* (24) *in place of f_l for all $0<\alpha'<\alpha$ and $0<\beta'<\beta$. The constant c there possibly depends on ω.*
(2) *The anisotropic sheet $B^{\alpha,\overline{\beta}}$ on $\mathbb{R}^{n+1}$, $0<\alpha,\ \beta_l$, $l=1,\dots,n$, $\overline{\beta}=(\beta_1,\dots,\beta_n)$, possesses a modification, again denoted by $B^{\alpha,\overline{\beta}}$, whose paths on $[a,b]\times K$ a.s. fulfil the Hölder conditions* (23) *and* (25) *in place of g for all $0<\alpha''<\alpha$ and $0<\beta_l''<\beta_l$.*

Lemma 6.1 can be combined with Lemma 5.1 by a simple smooth cut-off w.r.t. the $\mathbb{R}^n$-variable x: Choose K such that it contains an open neighbourhood of $\overline{D}$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be such that it is supported in K, $0 \leq \varphi(x) \leq 1$, $x \in \mathbb{R}^n$ and $\varphi(x) = 1$ for $x \in \overline{D}$.

Obviously the product $\varphi B^{\alpha,\beta}(\omega)$ still satisfies the mentioned Hölder conditions for $\mathbb{P}$-a.e. $\omega \in \Omega$.

This allows to state the following pathwise result.

COROLLARY 6.2 *Let the vector $B = (B^{\alpha,\beta_1}, \dots, B^{\alpha,\beta_n})$ consist of n hybrid fractional Brownian sheets B^{α,β_l} over $(\Omega, F, \mathbb{P})$ with $0 < \alpha$, $\beta_l < 1$, $l = 1, \dots, n$. Suppose $B^{\alpha',\overline{\beta'}}$ is an anisotropic fractional Brownian sheet over the same probability space, such that $\alpha' > 1 - \alpha$ and $\beta_l' > 1 - \beta_l$, $l = 1, \dots, n$.*

Then the integral

$$\int\int_{(a,b)\times D} \left\langle B, \left(\frac{\partial}{\partial t}\nabla\right)^+ B^{\alpha',\overline{\beta'}} \right\rangle d(t,x) \tag{31}$$

exists $\mathbb{P}$-a.s.

Proof Choose a smooth function φ as above and some p such that $\beta_l < 1/p$ for all $l = 1, \dots, n$. Then for a.e. ω, $\varphi B(\omega) = (\varphi B^{\alpha,\beta_1}(\omega), \dots, \varphi B^{\alpha,\beta_n}(\omega))$ and $\varphi B^{\alpha',\overline{\beta'}}(\omega)$ in place of f, and g respectively, fulfil the hypotheses of Definition 5.2 by Lemma 5.1 and the above discussion. For $\mathbb{P}$-a.e. $\omega \in \Omega$, the existence of

$$\int\int_{(a,b)\times D} \left\langle \varphi B(\omega), \left(\frac{\partial}{\partial t}\nabla\right)^+ B^{\alpha',\overline{\beta'}}(\omega) \right\rangle d(t,x)$$

follows. The a.s. existence of the integral (31) follows, its independence of the choice of φ is obvious. ■

Similar arguments yield a correspond result for integrals over D only.

7. Stochastic integrals

We consider stochastic versions of the integrals (18) and (28), where the integrators are given by the n-parameter Brownian motion and the hybrid Brownian sheet, respectively. For bounded, continuous and predictable integrands H they will be shown to coincide with corresponding integrals of (partial) Itô type.

Again $(\Omega, F, \mathbb{P})$ is a given probability space. We assume all processes to be real-valued, the extension to complex-valued is straightforward. Throughout this section, D denotes a bounded and convex domain in $\mathbb{R}^n$. We remark that under some conditions on the boundary, more general domains can be considered. We use the identification $x = (x_l', x_l)$, where $x_l' = (x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n)$. For any fixed $l = 1, \dots, n$ and $x_l' \in \mathbb{R}^{n-1}$, set

$$d_l^+(x_l') := \inf\{r \in \mathbb{R} : (x_l', re_l) \in \partial D\},$$

$$d_l^-(x_l') := \inf\{r > d_l^+(x_l') : (x_l', re_l) \in \partial D\}$$

and $\gamma_l^{\pm}(x_l') := d_l^{\pm}(x_l')e_l$. Finally, put

$$P_l(D) := \{x_l' \in \mathbb{R}^{n-1} : \exists x_l \in \mathbb{R}^n \text{ such that } (x_l', x_l) \in D\}.$$

7.1. *(Partial) stochastic integrals on D*

As mentioned, setting $\alpha = 1/2$ in (29), we arrive at the n-parameter Brownian motion W on $\mathbb{R}^n$. From the covariance structure it follows that for $l = 1, \ldots, n$ and $x_l' \in P_l$ fixed, $\{W^D(x_l', x_l) : x_l \in \mathbb{R}\}$ with

$$W^D(x_l', x_l) := W(x_l', x_l) - W(x_l', \gamma_l^+(x_l'))$$

is a Brownian motion on $\mathbb{R}$, $(x_l', \gamma_l^+(x_l'))$ playing the role of the origin.

For $l = 1, \ldots, n$ and $x_l' \in P_l$ fixed set

$$\mathcal{F}_{x_l}^{x_l'} := \sigma\big(W^D(x_l', y_l) : d_l^+(x_l') \le y_l \le x_l\big).$$

Let $\mathcal{P}^{x_l'}$ denote the σ-field on $[d_l^+(x_l'), \infty) \times \Omega$ generated by integrands of form

$$h(\omega, x_l) = X(\omega)\mathbf{1}_{(y_l, z_l]}(x_l) \quad \text{or} \quad h_0(\omega, x_l) = X_0(\omega)\mathbf{1}_{\{d_l^+(x_l')\}}(x_l),$$

where $\omega \in \Omega$, X is $\mathcal{F}_{y_l}^{x_l'}$-measurable, X_0 is $\mathcal{F}_{d_l^+(x_l')}^{x_l'}$-measurable and $d_l^+(x_l') \le y_l < z_l$. We set $\mathcal{P}^l := \cap_{x_l' \in P_l} \mathcal{P}^{x_l'}$ and call a random field $H = (H_1, \ldots, H_n)$ on $\mathbb{R}^n$ *predictable w.r.t. D* if H_l is $\mathcal{P}^l$-measurable for all $l = 1, \ldots, n$. For our purposes it seems convenient to assume that $H = (H_1, \ldots, H_n)$ is defined on the whole of $\mathbb{R}^n$. H is called *square integrable* if for all $l = 1, \ldots, n$ and all $x_l' \in P_l$, $\mathbb{E}\{\int_{d_l^+(x_l')}^{d_l^-(x_l')} H_l(x_l', x_l)^2 \mathrm{d}x_l\} < \infty$. For predictable and square integrable H, each integral

$$\int_{[d_l^+(x_l') d_l^-(x_l')]} H_l(x_l', x_l)\mathrm{d}W^D(x_l', x_l)\ ,\ l = 1, \ldots, n\ ,\ x_l' \in P_l,$$

is well defined in the Itô sense and fulfils a corresponding Itô isometry. We obtain a (partial) Itô-type integral for such integrands on D by setting

$$\int_{\overline{D}} \langle H, \mathrm{d}W\rangle := \sum_{l=1}^{n} \int_{P_l} \int_{[d_l^+(x_l'),\, d_l^-(x_l')]} H_l(x_l', x_l)\mathrm{d}W_j^D(x_l', x_l)\mathrm{d}x_l'. \tag{32}$$

Assuming in addition that H is bounded and a.s. continuous on $\partial D \subset \mathbb{R}^n$, Proposition 1.1 in [12] shows that the forward integral equals the Itô integral, in our situation that means

$$\begin{aligned}\int_{[d_l^+(x_l'),\, d_l^-(x_l')]} & H_l(x_l', x_l)\mathrm{d}W^D(x_l', x_l) \\ &= \lim_{r\to 0} \frac{1}{r} \int_{d_l^+(x_l')}^{d_l^-(x_l')} H_l(x_l', x_l)\big(W^D(x_l', x_l + re_l) - W^D(x_l', x_l)\big)\mathrm{d}x_l,\end{aligned} \tag{33}$$

where the limit is taken in the mean square. The correction denoted by the superscript D may be omitted in the difference, and with the notation of the former sections we consider

$$\int_{\overline{D}} \langle H, \nabla_{\mathbb{P}}^+ W\rangle\, \mathrm{d}x := \lim_{r\to 0} \int_D \langle H, \nabla_r^+ W\rangle \mathrm{d}x, \tag{34}$$

the limit taken in square mean.

LEMMA 7.1 *Let W denote the n-parameter Brownian motion on $\mathbb{R}^n$ and let $H=(H_1,\ldots,H_n)$ be a bounded random field on $\mathbb{R}^n$, predictable w.r.t. D and a.s. continuous on $\partial D\subset\mathbb{R}^n$. Then the limit* (34) *exists and equals the Itô-type integral* (32),

$$\int_{\overline{D}}\langle H,\nabla_{\mathbb{P}}^{+}W\rangle\,\mathrm{d}x=\int_{\overline{D}}\langle H,\mathrm{d}W\rangle.$$

7.2. *Stochastic integrals on* $[a,b]\times\overline{D}$

Setting $\alpha=\beta=1/2$ in (30), we obtain a Gaussian field $W=\{W(t,x)\colon t\in\mathbb{R},x\in\mathbb{R}^n\}$, which might be called the *hybrid Brownian sheet* on $\mathbb{R}^{n+1}$. A corresponding (partial) Itô-type integral on $[a,b]\times\overline{D}$ can be defined following the construction of the Itô integral in the plane, see e.g. [4,17,18]:

For s, $t\in\mathbb{R}$ and x, $y\in\mathbb{R}^n$, we write $(t,x)\prec(s,y)$ if and only if $t\leq s$ and $x_l\leq y_l$, $l=1,\ldots,n$. We write $(t,x)\prec\prec(s,y)$ if all inequalities are strict.

Fix some l and $x_l'\in P_l$ as above, then $\{W^D(t,x_l'.x_l)\colon t\in\mathbb{R},x_l\in\mathbb{R}\}$ with

$$W^D(t,x_l',x_l):=W(t,x_l',x_l)-W(t,x_l',\gamma_l^+(x_l'))$$

is a Brownian sheet on $\mathbb{R}^2$ and $(0,x_l',\gamma_l^+(x_l'))$ plays the role of the origin. With l and $x_l'\in P_l$ still fixed we set

$$\mathcal{F}_{t,x_l}^{x_l'}:=\sigma\big(W^D(t,x_l',y_l)\colon(0,d_l^+(x_l'))\prec(s,y_l)\prec(t,x_l)\big).$$

Now let $\mathcal{P}^{x_l'}$ denote the σ-field on $[a,\infty)\times[d_l^+(x_l'),\infty)\times\Omega$ generated by integrands of form

$$h(\omega,t,x_l)=X(\omega)\mathbf{1}_{(c,d]}(t)\mathbf{1}_{(y_l,z_l]}(x_l),\tag{35}$$

where $\omega\in\Omega$, X is $\mathcal{F}_{(c;y_l)}^{x_l'}$-measurable, $(a,d_l^+(x_l'))\prec(c,y_l)\prec\prec(z_l,d)$, together with integrands of a similar form but with $\mathbf{1}_{\{0\}}(t)$ in place of $\mathbf{1}_{(c,d]}(t)$ or $\mathbf{1}_{\{d_l^+(x_l')\}}(x_l)$ in place of $\mathbf{1}_{(y_l,z_l]}(x_l)$ or both, considered under the respective measurability assumptions. Put again $\mathcal{P}^l:=\cap_{x_l'\in P_l}\mathcal{P}^{x_l'}$ and call a random field $H=(H_1,\ldots,H_n)$ on $\mathbb{R}^{n+1}$ *predictable w.r.t.* $(a,b)\times D$ if H_l is $\mathcal{P}^l$-measurable for all $l=1,\ldots,n$. H is called *square integrable* if for all $l=1,\ldots,n$ and all $x_l'\in P_l$, $\mathbb{E}\{\int\int_{(a,b)\times(d_l^+(x_l'),d_l^-(x_l'))}H_l(t,x_l',x_l)^2d(t,x_l)\}<\infty$. For predictable and square integrable H, each integral

$$\int\int_{[a,b]\times[d_l^+(x_l'),d_l^-(x_l')]}H_l(t,x_l',x_l)\mathrm{d}W(t,x_l',x_l)\tag{36}$$

is well defined as two-parameter Itô integral (of the first kind). For integrands of form (35) it equals $X\Delta_{d-c,z_l-y_l}W(c,x_l',y_l)$, the difference according to (21) referring to c and y_l. The usual isometry property holds at the level of (36). Setting

$$\int\int_{[a,b]\times\overline{D}}\langle H,\mathrm{d}W\rangle:=\sum_{l=1}^{n}\int_{P_l}\int\int_{[a,b]\times[d_l^+(x_l'),d_l^-(x_l')]}H_l(t,x_l',x_l)\mathrm{d}W(t,x_l',x_l)\mathrm{d}x_l',\tag{37}$$

we obtain an integral of (partial) Itô type.

Now consider the limit

$$\lim_{|(u,r)|\to 0} \int_a^b \int_D \left\langle H(t,x), \nabla_r^+ \frac{W_b(t+u,x) - W_b(t,x)}{ur} \right\rangle \mathrm{d}x\,\mathrm{d}t\ , \tag{38}$$

taken in the mean square and, whenever it exists. The correction $W_b(t,x) = \mathbf{1}_{(a,b)}(t) \times (W(t,x) - W(b,x))$ w.r.t. (a,b) is understood pathwise. For u, $r > 0$ fixed a member of the sequence in (38) equals

$$\sum_{l=1}^{n} \int\int_{(a,b)\times D} H_l(t,x) \frac{\Delta_{u,re_l} W_b(t;x)}{ur} \mathrm{d}t\,\mathrm{d}x\ ,$$

with $\Delta_{u,re_l} W_b(t,x)$ according to (21). Consider its average version

$$\begin{aligned}(A)&\int_{(a,b)\times D} \left\langle H, \left(\frac{\partial}{\partial t}\nabla\right)_{\mathbb{P}}^{+} W\right\rangle d(t,x)\\ &:= \lim_{|(\varepsilon,r)|\to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_a^b \int_D \left\langle H(t), \nabla_r^+ \frac{W_b(t+u) - W_b(t)}{u} \right\rangle \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}u,\end{aligned} \tag{39}$$

taken in square mean, whenever it exists. This is the *stochastic variant of the pathwise* (*hybrid*) *average integral* (28). Adapting the proofs known from the one-parameter case, [12,16], we see that for suitable integrands, it equals the Itô-type integral:

LEMMA 7.2 *Let* $H = (H_1, \ldots, H_n)$ *be a random field on* $\mathbb{R}^{n+1}$, *predictable w.r.t.* $[a,b] \times \overline{D}$, *a.s. bounded and continuous. Then*:

(i) *If limit* (38) *exists, so does the average limit* (39), *and both agree*.
(ii) *The average limit* (38) *exists and equals the Itô-type integral* (37).

Consequently also

$$(A)\int_{(a,b)\times D} \left\langle H, \left(\frac{\partial}{\partial t}\nabla\right)_{\mathbb{P}}^{+} W\right\rangle d(t,x) = \int\int_{[a,b]\times \overline{D}} \langle H, \mathrm{d}W\rangle.$$

(i) is an obvious modification of arguments from [16, p. 3]. For convenience, the proof of (ii) is sketched in Appendix 1 for the case $n = 1$, $D = (a', b')$, where W^D is the Brownian sheet on $\mathbb{R}^2$.

We finally remark that in the case $n = 1$, [4] used a weakly adapted two-parameter integral to study partial differential equations, contained in a construction similar to the above using the filtrations $\mathcal{F}_t^{x_l'} := \bigvee_{d_l^+(x_l') \le x_l} \mathcal{F}_{t,x_l}^{x_l'}$. The above can easily be adapted to this case.

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Appendix 1. Proofs

Proof of Proposition 4.4 Given $a \in \mathbb{R}$, the translation T_a is defined by $(f \circ T_a)(x) := f(x + a)$.

Proof The estimate for the pairing is obvious, we need to verify existence and value of the forward integral. It suffices to consider a single summand in (17). We fix an integer $1 \le l \le n$ and write α for α_l. For $r > 0$ set

$$I_l(f, g, r) = \int_{\mathbb{R}^n} \mathbf{1}_D f_l \partial_{l,r}^+ g \, \mathrm{d}x = \int_{\mathbb{R}^n} \mathbf{1}_D f_l(x) \frac{g \circ T_{re_l} - g}{r}(x) \, \mathrm{d}x.$$

First assume $g \in \mathcal{S}(\mathbb{R}^n)$. Then

$$I_l(f, g, r) = \int_0^1 \int_{\mathbb{R}^n} (\mathbf{1}_D f_l)^\wedge(\xi) \, e^{irt\xi_l} i\xi_l \, \hat{g}(\xi) \, \mathrm{d}\xi \, \mathrm{d}t.$$

For fixed $t > 0$, the inner integral in the last line equals

$$\int_{\mathbb{R}^n} \big((1 + \xi_l^2)^{\alpha/2} (\mathbf{1}_D f_l)^\wedge\big)^\vee(x) \big(i\xi_l(1 + \xi_l^2)^{-\alpha/2} \;\; e^{irt\xi_l e_l} \, \hat{g}\big)^\vee(x) \, \mathrm{d}x$$

and by Hölder's inequality this is (componentwise) bounded above by

$$\begin{aligned} &\Big\| \big((1 + \xi_l^2)^{\alpha/2} (\mathbf{1}_D f_l)^\wedge\big)^\vee | L_p(\mathbb{R}^n) \Big\| \Big\| \big(i\xi_l(1 + \xi_l^2)^{-\alpha/2} e^{irt\xi_l} \hat{g}\big)^\vee | L_{p'}(\mathbb{R}^n) \Big\| \\ &\le c \Big\| \big((1 + \xi^2)^{\alpha/2} (\mathbf{1}_D f_l)^\wedge\big)^\vee | L_p(\mathbb{R}^n) \Big\| \Big\| \big((1 + \xi_l^2)^{(1-\alpha)/2} \hat{g}\big)^\vee \circ T_{rte_l} | L_{p'}(\mathbb{R}^n) \Big\|, \end{aligned}$$

note that $(1 + \xi_l^2)^{\alpha/2}(1 + \xi^2)^{-\alpha/2}$ and $\xi_l(1 + \xi_l^2)^{-1/2}$ are Fourier multipliers. Hence by Lemma 4.3 and translation invariance of the $L_{p'}(\mathbb{R}^n)$-norm,

$$\big| I_l(f, g, r) \big| \le c \Big\| f_l | H_p^\alpha(\mathbb{R}^n) \| \| g | H_{p',l}^{1-\alpha}(\mathbb{R}^n) \Big\|. \tag{A1}$$

For fixed $0 < r < r'$ we similarly obtain

$$\begin{aligned} \big| I_l(f, g, r) - I_l(f, g, r') \big| &\le c \Big\| f_l | H_p^\alpha(\mathbb{R}^n) \Big\| \\ &\times \int_0^1 \Big\| \big((1 + \xi_l^2)^{(1-\alpha)/2} \hat{g}\big)^\vee \circ T_{tre_l} - \big((1 + \xi_l^2)^{(1-\alpha)/2} \hat{g}\big)^\vee \circ T_{tr'e_l} | L_{p'}(\mathbb{R}^n) \Big\| \mathrm{d}t. \end{aligned} \tag{A2}$$

Approximating $g \in H_{p',l}^{1-\alpha}(\mathbb{R}^n)$ by functions $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ in $H_{p',l}^{1-\alpha}(\mathbb{R}^n)$, (A1) and (A2) carry over, notice that by Hölder

$$\big| I_l(f, g - \varphi_j, r, w) \big| \le \| f_l | L_p(\mathbb{R}^n) \| \left\| \frac{g \circ T_{re_l} - g}{r} - \frac{\varphi_j \circ T_{re_l} - \varphi_j}{r} | L_{p'}(\mathbb{R}^n) \right\|$$

and φ_j tends to g in $L_{p'}(\mathbb{R}^n)$.

Finally, the right-hand side of (A2) tends to zero as r and r' do, since for any $u \in L_{p'}(\mathbb{R}^n)$ and $h \in \mathbb{R}$, $\lim_{r \to 0} \| u - u(\cdot + rh) | L_{p'}(\mathbb{R}^n) \| = 0$. Therefore (16) exists. On the other hand, for $f \in H_p^\alpha(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$

$$\left| \left\langle \mathbf{1}_D f_l, \frac{\partial g}{\partial x_l} \right\rangle - I_l(f, g, r) \right| \le c \Big\| f_l | H_p^\alpha(\mathbb{R}^n)^{\|} \int_0^1 \Big\| \big(i\xi_l(1 + \xi_l^2)^{-\alpha/2} \hat{g}\big)^\vee - \big(i\xi_l(1 + \xi_l^2)^{-\alpha/2} \hat{g}\big)^\vee \circ T_{tre_l} | L_{p'}(\mathbb{R}^n) \Big\| \mathrm{d}t$$

tends to zero as r does by similar arguments, and completion yields the desired equality. ■

Proof of Lemma 5.7 For $n=1$ and $D=(a',b')$. In this case the assertion reads

$$\frac{1-\varepsilon}{\Gamma(\varepsilon)}\int_0^\infty u^{\varepsilon-1}\int_a^b\int_{a'}^{b'} f(t,x),\ \frac{\Delta_{u,r}g_{b-}(t,x)}{ur}\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}u$$
$$=(-1)^\alpha\int_a^b\int_{a'}^{b'} D_{a+}^{\alpha-\varepsilon}f(t,x),\ \frac{D_{b-}^{1-\alpha}g_{b-}(t,x+r)-D_{b-}^{1-\alpha}g_{b-}(t,x)}{r}\mathrm{d}x\,\mathrm{d}t. \qquad \text{(A3)}$$

Proof For u and r fixed, we may rewrite the inner integrals on the left-hand side of (A3) as

$$\int_a^b\int_{a'}^{b'} f(t,x)\frac{\Delta_{u,r}g_{b-}(t,x)}{ur}\mathrm{d}x\,\mathrm{d}t$$
$$=(-1)^{\alpha-\varepsilon}\int_a^b\int_{a'}^{b'} D_{a+}^{\alpha-\varepsilon}f(t,x)\frac{I_{b-}^{\alpha-\varepsilon}\Delta_{u,r}g_{b-}(t,x)}{ur}\mathrm{d}x\,\mathrm{d}t, \qquad \text{(A4)}$$

as Fubini's theorem and the integration-by-parts formula for fractional derivatives (5) show. Next, note that for $r>0$ fixed,

$$\frac{(1-\varepsilon)(-1)^{1-\varepsilon}}{\Gamma(\varepsilon)}\int_\delta^\infty \frac{I_{b-}^{\alpha-\varepsilon}\Delta_{u,r}g_{b-}(t,x)}{u^{2-\varepsilon}r}\,\mathrm{d}u,$$

seen as Marchaud derivative, converges to

$$-\frac{D_{b-}^{1-\alpha}g_{b-}(t,x+r)-D_{b-}^{1-\alpha}g_{b-}(t,x)}{r} \qquad \text{(A5)}$$

as δ decreases to zero. By Fubini's theorem and Hölder's inequality we may therefore rewrite the left-hand side of (A3) as

$$(-1)^\alpha\int_a^b\int_{a'}^{b'} D_{a+}^{\alpha-\varepsilon}f(t,x),\ \frac{D_{b-}^{1-\alpha}g_{b-}(t,x+r)-D_{b-}^{1-\alpha}g_{b-}(t,x)}{r}\mathrm{d}x\,\mathrm{d}t,$$

as desired. ■

Proof of Lemma 5.1

Proof We verify (II) for f. In the latter case multiply $g(b-)$ by a smooth cut-off function of x, which equals one on a neighbourhood of $\overline{D}$.

By standard embedding theorems, $\|h|H_p^{\beta'}(\mathbb{R})\|$ is bounded by a constant times

$$\|h|L_p(\mathbb{R})\|+\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|h(x+y)-h(x)|^p}{|y|^{1+\beta' p}}\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \qquad \text{(A6)}$$

in case $p\le 2$. For $p>2$ this has to be replaced by

$$\|h|L_p(\mathbb{R})\|+\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|h(x+y)-h(x)|^p\mathrm{d}x\right)^{2/p}\frac{\mathrm{d}y}{|y|^{1+2\beta'}}\right)^{1/2}, \qquad \text{(A7)}$$

see e.g. [29]. For compactly supported h, Hölder's inequality quickly shows that (A7) is bounded by a constant times (A6) with $\beta'+\delta$ in place of β' for any $\delta>0$. Hence it suffices to show that the $L_p((a,b))$-norm of (A6), with

$$D_{a+}^{\alpha'}f(t,x)=c_{\alpha'}\mathbf{1}_{(a,b)}(t)\left(\frac{f(t,x)}{(t-a)^{\alpha'}}+\alpha'\int_a^t\frac{f(t,x)-f(\tau,x)}{(t-\tau)^{\alpha'+1}}\mathrm{d}\tau\right) \qquad \text{(A8)}$$

in place of h, is finite. An estimate of the first summand of (A8) together with the difference part of (A6) in $L_p((a,b))$ is given by

$$\int_a^b \int\int_{|x-y|<r_0} \left(\int_t^a \frac{|f(t,x)-f(\tau,x)-f(t,y)+f(\tau,y)|}{(t,\tau)^{1+\alpha'}} \mathrm{d}\tau\right)^p \times \frac{1}{|x-y|^{1+\beta' p}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$\leq \int_a^b \int\int_{|x-y|<r_0} \left(\int_a^t |t-\tau|^{(\alpha-\alpha')-1} \mathrm{d}\tau\right)^p |x-y|^{(\beta-\beta')p-1} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t < \infty,$$

we have used (22). The terms arising from combinations of the remaining summands of (A8) and (A6) obey similar estimates obtained from (22) and (24) and the fact that each $f(t,\cdot)$ has compact support.

(I) follows from the continuity of f at a, seen as $H_p^{\alpha'}(\mathbb{R})$-valued function, which is shown by arguments similar to the above.

For g one can proceed similarly, it suffices to note that with $l=1,\ldots,n$ fixed, $\mathbb{R}_l = \operatorname{span}\{e_l\}$, and $\mathbb{R}_l^\perp \subset \mathbb{R}^n$ denoting its orthogonal complement,

$$\left\| g | H^{\beta_l}_{p',l}(\mathbb{R}^n)\right\| = \left\| \left((1+\xi_l^2)^{\beta_l/2}\hat{g}\right)^\vee | L_{p'}(\mathbb{R}^n)\right\|$$
$$= \left(\int_{\mathbb{R}_l^\perp} \left\| g(\xi_l', \cdot) | H^{\beta_l}_{p'}(\mathbb{R})\right\|^{p'} \mathrm{d}\xi_l'\right)^{1/p'},$$

where $\xi_l' = (\xi_1,\ldots,\xi_{l-1},\xi_{l+1},\ldots,\xi_n)$ as before. ■

Proof of Lemma 7.2 (ii) Assume $n=1$, $D=(a',b')$ in Lemma 7.2, then W^D is the Brownian sheet.

Proof Notice first that if (38) exists with $W_{b,b'}(t,x) = W(t,x) - W(b,x) - W(t,b') + W(b,b')$ in place of $W_b(t,x)$, it exists in its original form. This follows using bounded convergence. Now follow [12]: For any u, $u>0$ and any $(t,x)\in[a,b]\times[a'.b']$,

$$\Delta_{u,r} W_{b,b'}(t,x) = W((t+u)\wedge b, (x+r)\wedge b') - W(t,(x+r)\wedge b') - W((t+u)\wedge b, x) + W(t,x).$$

Now it follows that with $Y(t,x) := H(t,x)\mathbf{1}_{\{(t,x)\prec\prec(s,y)\prec(t+u,x+r)\}}$,

$$\int\int_{[a,b]\times[a',b']} Y\mathrm{d}W = H(t,x)\Delta_{u,r}W_{b,b'}(t,x),$$

the right-hand side in the Itô sense. By Lemma A.1 below then

$$\frac{1}{ur}\int_{[a,b]\times[a',b']} H(t,x)\Delta_{u,r}W_{b,b'}(t,x)\mathrm{d}(t,x) = \int\int_{[a,b]\times[a',b']}$$
$$\times \left(\frac{1}{ur}\int_{[s-u,s]\times[y-r,y]} H(t,x)\mathbf{1}_{[a,b]\times[a',b']}(t,x)\mathrm{d}(t,x)\right)\mathbf{1}_{[a,b]\times[a',b']}(s,y)\mathrm{d}W(s,y).$$

By the isometry, the expectation of the square of

$$\int\int_{[a,b]\times[a',b']} \left(\frac{1}{ur}\int\int_{[s-u,s]\times[y-r,y]} H(t,x)\mathbf{1}_{[a,b]\times[a',b']}(t,x)d(t,x) - H(s,y)\right)\mathrm{d}W(s,y)$$

equals the expectation of

$$\int\int_{[a,b]\times[a',b']} \left(\frac{1}{ur}\int\int_{[s-u,s]\times[y-r,y]} H(t,x)\mathbf{1}_{[a,b]\times[a',b']}(t,x)d(t,x) - H(s,y)\right)^2 \mathrm{d}(s,y).$$

Now take into account that along any increasing path (u,r),

$$\lim_{|(u,r)|\to 0}\frac{1}{ur}\int_{[s-u,s]\times[y-r,y]} H(t,x)\mathbf{1}_{[a,b]\times[a',b']}(t,x)\,d(t,x)=H(s,y)\text{ a.s.}$$

by the a.s. continuity of H. ■

LEMMA A.1 *Let H be a bounded and $\mathcal{P}\otimes\mathcal{B}([a,b]\times[a',b'])$-measurable mapping on $\Omega\times[a,b]\times[a',b']\times[a,b]\times[a',b']$. Then*

$$\int_{[a,b]\times[a',b']}\int_{[a,b]\times[a',b']} H(u,v)\mathrm{d}W(u)\mathrm{d}v=\int_{[a,b]\times[a',b']}\int_{[a,b]\times[a',b']} H(u,v)\mathrm{d}v\,\mathrm{d}W(u).$$

Here $\mathcal{B}([a,b]\times[a',b'])$ denotes the Borel σ-field over $([a,b]\times[a',b'])$. For integrands of form (35) this is obvious, for arbitrary H it follows by a monotone class argument, use the isometry together with the bounded convergence theorem.

Appendix 2. Riemann–Stieltjes integrals in the sense of Gowurin

The following construction had been introduced in [19]. Let E and F be real Banach spaces normed by $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively, and $L(E,F)$ the space of bounded linear operators from E into F, endowed with the operator norm. Let $(a,b)\subset\mathbb{R}$ be a bounded interval and consider two functions $f\colon (a,b)\to E$ and $U\colon (a,b)\to L(E,F)$. We assume throughout that the limits $f(a+)$, $f(b-)$, $U(a+)$ and $U(b-)$ exist in the respective spaces and set $f(a):=f(a+)$, etc. below. Let $\mathcal{P}_\Delta=\{t_i\colon\ i=0,\ldots,k, a=t_0<t_1<\cdots<t_k=b\}$ with some $k\in\mathbb{N}$ be a partition of (a,b) with $\max_i|t_i-t_{i-1}|<\Delta$. If the Riemann–Stieltjes sums

$$S(f,U,\mathcal{P}_\Delta)=\sum_{i=1}^{k}[(U(t_i)-U(t_{i-1}))f(\tau_i)],$$

where $\tau_i\in[t_{i-1},t_i]$, converge to a limit along a sequence of refining partitions $\mathcal{P}_\Delta$ of the above type as Δ goes to zero, this limit is called *Riemann–Stieltjes integral* (*in the sense of Gowurin*) and denoted by

$$(\mathrm{RS})\int_a^b f\,\mathrm{d}U:=\lim_{\Delta\to 0}S(f,U,\mathcal{P}_\Delta)\ .$$

A function $U\colon(a,b)\to L(E,F)$ is said to have the *ω-property on* (a,b), if there exists some $M>0$ such that for any partition $\mathcal{P}$ of the above type and any $x_i\in E$, $i=0,\ldots,k-1$, $\|\sum_{i=1}^k(U(t_i)-U(t_{i-1}))x_i\|_F\le M\max_i\|x_i\|_E$. Similar to the scalar-valued case, one can show that $(\mathrm{RS})\int_a^b f\mathrm{d}U$ exists if f is strongly continuous and U has the ω-property. It also exists if U is strongly continuous and f is of *bounded variation on* (a,b), i.e. there exists some $M>0$ such that $\sup_{\mathcal{P}}\sum_{i=1}^k\|f(t_i)-f(t_{i-1})\|_E\le M$, the supremum taken over all partitions of (a,b).

The concept extends to complex Banach spaces E and F in the coordinatewise sense. The following simple lemma holds.

LEMMA A.2 *Suppose U: $(a,b)\to L(E,F)$ is uniformly bounded on (a,b) and has the ω-property. Assume Φ: $(a,b)\times(a,b)\to E$ is a function such that for a.e. $\tau\in(a,b)$, $\Phi(\cdot,\tau)$ is strongly continuous on $(a,b)\backslash\{\tau\}$ and*

$$\sup_{t\in(a,b)}\int_a^b\left\|\Phi(t,\tau)\right\|_E\mathrm{d}\tau<\infty\ . \tag{A9}$$

Then both integrals below exist and

$$(\mathrm{RS})\int_a^b \int_a^t \Phi(t,\tau)\mathrm{d}\tau\,\mathrm{d}U(t) = \int_a^b (\mathrm{RS})\int_\tau^b \Phi(t,\tau)\mathrm{d}U(t)\mathrm{d}\tau.$$

Proof By (A9), $\Psi(t) = \int_a^t \Phi(t,\tau)\mathrm{d}\tau$ is strongly continuous on (a,b) and with $\tau_i \in [t_{i-1}, t_i]$, the right-hand side rewrites

$$\begin{aligned}\int_a^b \Psi(t)\mathrm{d}U(t) &= \lim_{\Delta\to 0}\sum_{i=1}^k \int_a^{\tau_i} (U(t_i) - U(t_{i-1}))\Phi(\tau_i,\tau)\mathrm{d}\tau \\ &= \lim_{\Delta\to 0}\int_a^b \sum_{i=1}^k (U(t_i) - U(t_{i-1}))\mathbf{1}_{[\tau+\varepsilon,b)}(\tau_i)\Phi(\tau_i,\tau)\mathrm{d}\tau \\ &\quad + \lim_{\Delta\to 0}\int_a^b \sum_{i=1}^k (U(t_i) - U(t_{i-1}))\mathbf{1}_{(\tau,\tau+\varepsilon)}(\tau_i)\Phi(\tau_i,\tau)\mathrm{d}\tau\end{aligned} \tag{A10}$$

for any $\varepsilon > 0$. We have used the uniform boundedness of U in the first equality. By the ω-property of U, for any $a \le c < d \le b$,

$$\left\| \sum_{i=1}^k (U(t_i) - U(t_{i-1}))\mathbf{1}_{(c,d)}(\tau_i)\Phi(\tau_i,\tau)\right\|_F \le M \max_i \mathbf{1}_{(c,d)}(\tau_i)\left\|\Phi(\tau_i,\tau)\right\|_F$$

and by (A9),

$$\begin{aligned}&\int_a^b \left\| \sum_{i=1}^k (U(t_i) - U(t_{i-1}))\mathbf{1}_{(c,d)}(\tau_i)\Phi(\tau_i,\tau)\right\|_F \mathrm{d}\tau \\ &\le M \max_i \int_a^b \mathbf{1}_{(c,d)}(\tau_i)\left\|\Phi(\tau_i,\tau)\right\|_E \mathrm{d}\tau < \infty\ .\end{aligned}$$

Therefore the first summand in (A10) is $\int_a^b \int_{\tau+\varepsilon}^b \Phi(t,\tau)\mathrm{d}U(t)\mathrm{d}\tau$, while the second is bounded by $M \sup_{t\in(a,b)} \int_{(t-\varepsilon)\vee a}^t \|\Phi(t,\tau)\|_E \mathrm{d}\tau$, which tends to zero as ε does. ■

With the aid of this lemma it can be seen that for suitable functions f and U the integral according to Definition 3.1 coincides with the Riemann–Stieltjes integral. Using Lemma A.2 one can follow the lines of [14, Theorem 2.4.(i)] to obtain:

PROPOSITION A.3 *Suppose* U: $(a,b) \to L(E,F)$ *has the* ω*-property and* f: $(a,b) \to E$ *is strongly continuous. Assume further*, f *and* U *satisfy the hypotheses of Definition* 3.1 *and with some* $0 \le \alpha \le 1$,

$$\sup_{t\in(a,b)} \int_a^b \left\| D_{a+}^\alpha f_{a+}(\tau)\right\|_E (t-\tau)^{\alpha-1}\mathrm{d}\tau < \infty\ .$$

Then $\int_a^b f\,\mathrm{d}U = (\mathrm{RS})\int_a^b f\,\mathrm{d}U$.