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Probability Theory and Related Fields

Integration with respect to fractal functions and stochastic calculus. I

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Abstract. The classical Lebesgue–Stieltjes integral $\int_a^b f\,dg$ of real or complex-valued functions on a finite interval (a, b) is extended to a large class of integrands f and integrators g of unbounded variation. The key is to use composition formulas and integration-by-part rules for fractional integrals and Weyl derivatives. In the special case of Hölder continuous functions f and g of summed order greater than 1 convergence of the corresponding Riemann–Stieltjes sums is proved.

The results are applied to stochastic integrals where g is replaced by the Wiener process and f by adapted as well as anticipating random functions. In the anticipating case we work within Slobodeckij spaces and introduce a stochastic integral for which the classical Itô formula remains valid. Moreover, this approach enables us to derive calculation rules for pathwise defined stochastic integrals with respect to fractional Brownian motion.

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0. Introduction

In order to motivate our paper we recall some well-known facts from Stieltjes integration.

Throughout the paper we consider Borel measurable real (or complex-valued) functions on $\mathbb{R}$, most often vanishing outside a given finite interval $[a, b]$.

If such a function g has finite variation on $[a,b]$ then it may be represented by $g = g_1 - g_2$ where g_1 and g_2 are monotone. Denote the finite Borel measures associated with g_1 and g_2 by μ_1 and μ_2, repectively. The *Lebesgue–Stieltjes integral* of a function f with respect to g is defined by

$$\text{(L-S)} \int_a^b f(x)\, dg(x) := \int_a^b f(x)\, d\mu_1(x) - \int_a^b f(x)\, d\mu_2(x) \qquad (1)$$

provided that f is Lebesgue integrable with respect to the variation measure $\mu := \mu_1 + \mu_2$ on $[a,b]$.

In the special case f being continuous this integral agrees with the *Riemann–Stieltjes integral* given by

$$\text{(R-S)} \int_a^b f(x)\, dg(x) := \lim_{\Delta \to 0} \sum_i f(x_i^*)(g(x_{i+1}) - g(x_i)) \qquad (2)$$

where convergence holds uniformly in all finite partitions $\mathscr{P}_\Delta := \{a =: x_0 \le x_0^* \le x_1 \le \cdots \le x_n \le x_n^* \le x_{n+1} = b\}$ with $\max_i |x_{i+1} x_i| < \Delta$. The assumption on g ensures the absolute convergence of the above Riemann–Stieltjes sums.

In general, the Riemann–Stieltjes integral of f with respect to g is determined if the uniform convergence in (2) holds (but not necessarily the absolute convergence). As a corollary of the Banach–Steinhaus theorem the following was shown: If for some g the convergence (2) holds for all continuous f then g must be of finite variation (see, e.g. [10]).

In stochastic calculus based on martingale theory the absolute convergence of the Riemann–Stieltjes sums is replaced by convergence in mean square or, more generally, in probability. In this approach g is a random process being a semimartingale. Again, one cannot choose arbitrary (random) continuous functions f as integrands unless g has finite variation. However, the class of square integrable adapted random functions provides an appropriate solution. In particular, if the Wiener process W plays the role of g one turns to classical Itô calculus. The so-called Skorohod integrals extend this construction to certain anticipating integrands f.

In the present paper we extend the Stieltjes integrals to functions of unbounded variation via fractional calculus. Recall that if f or g are smooth on (a,b) the Lebesgue–Stieltjes integral may be rewritten as

$$\text{(L-S)} \int_a^b f\, dg = \int_a^b f(x) g'(x)\, dx \qquad (3)$$

and

$$(\text{L-S})\int_a^b f\,dg = -\int_a^b f'(x)g(x)\,dx + f(b-)g(b-) - f(a+)g(a+) \quad (4)$$

respectively.

(Throughout the paper we denote $f(a+) := \lim_{\delta\searrow 0} f(a+\delta)$, $g(b-) := \lim_{\delta\searrow 0} g(b-\delta)$ supposing that the limits exist.) The main idea of our approach consists in replacing the ordinary derivatives by appropriate fractional derivatives in the sense of Riemann and Liouville and using their Weyl representation. We put

$$f_{a+}(x) := 1_{(a,b)}(x)(f(x) - f(a+)) \quad (5)$$

$$g_{b-}(x) := 1_{(a,b)}(x)(g(x) - g(b-)) \quad (6)$$

and define the integral by

$$\int_a^b f\,dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x)\,dx$$
$$+ f(a+)(g(b-) - g(a+))$$

for certain $0 \le \alpha \le 1$ provided that f and g satisfy some fractional differentiability conditions in L_p-spaces, where $(-1)^\alpha = e^{i\pi\alpha}$. (In the case of real-valued g the function $(-1)^\alpha D_{b-}^{1-\alpha} g_{b-}(x)$ is real-valued.)

The paper is organized as follows:

In section 1 some background from fractional calculus is summarized.

The integral mentioned above is introduced in Section 2. We show that for Hölder continuous g and step functions f the integral agrees with the corresponding Riemann–Stieltjes sums. Theorem 2.4 provides general conditions when our integral coincides with the Lebesgue–Stieltjes integral. The additivity of the integral as function of the boundary is proved in Theorem 2.6.

Because of the choice of left and right sided fractional derivatives the above integral seems to be directed forward. Therefore we construct in Section 3 a backward integral in a similar way. It turns out that both the integrals coincide. As a corollary we obtain an integration-by-part formula for these integrals.

The special case of Hölder continuous functions f and g of summed order greater than 1 is studied in Section 4. As a basic result we prove the convergence of the Riemann–Stieltjes sums (2) to our integral. This implies that the classical chain rule for the change of variables remains valid. We further prove that the integral as function of the boundary is Hölder continuous of the same order as g. This leads to an analogue to Lebesgue integration with respect to a measure which is absolutely continuous with respect to a reference measure via densities.

The second part of the paper, i.e. Section 5, deals with applications to stochastic calculus.

In Section 5.1 we demonstrate on the example of fractional Brownian motion B^H that our integral may be used in order to construct (stochastic) integrals for almost all realizations of stochastic processes without semimartingale properties. As long as we assume Hölder continuity (or fractional differentiability) of the random integrands f of order greater than one minus that of the integrator we do not need any adaptedness. (This makes it possible to investigate stochastic differential equations with respect to fractional Brownian motion of order greater than 1/2.)

In Section 5.2 we replace g by the Wiener process W and show that for adapted random L_2-functions f of "fractional degree of differentiability" greater than 1/2 our integral agrees with the classical Itô integral. For the more general class of functions having fractional derivatives in some L_2-sense of all orders less than 1/2 we prove convergence in probability of the integrals

$$\mathbf{I}^{1-\epsilon}f = (-1)^{1/2-\epsilon/2} \int_a^b D_{a+}^{1/2-\epsilon/2} f(x) D_{b-}^{1/2-\epsilon/2} W_{b-}(x)\, dx$$

to the Itô integral $\mathbf{I}f$ as $\epsilon \searrow 0$. Sufficient conditions for mean square convergence are also provided.

Finally, in section 5.3 we extend these results to anticipating random functions f. We first define the anticipating integral

$$(A) \int_0^1 f\, dW = \sum_{n=0}^{\infty} \left(\widetilde{\mathbf{I}}^{n+1} f^n + n \int_0^1 \widetilde{\mathbf{I}}^{n-1} f^n(\cdot, t, t-)\, dt \right)$$

in terms of the Itô-Wiener chaos expansion $f = \sum_0^\infty \widetilde{\mathbf{I}}^n f^n$. For adapted f this integral coincides with the Itô integral. In the anticipating case we introduce the Slobodecki-type spaces $\mathbf{W}_{2,+}^{\alpha}(0,1)$ of random functions f and show that for $\alpha > 1/2$ (where the integral agrees with the extended Stratonovitch integral $\int_0^1 f \circ\, dW$) it is equal to the fractional integral $\int_0^1 f\, dW$ considered before. If $f \in \mathbf{W}_{2,+}^{\alpha}$ for any $\alpha < 1/2$ and the above integrals $\mathbf{I}^{1-\epsilon} f$ converge in the mean square then the limit agrees with $(A) \int_0^1 f\, dW$. A sufficient condition for this convergence is that f lies additionally in the space $\mathbb{L}_C^{1,2}$ which is introduced in the theory of Skorohod integrals $\delta(f)$. For such f we obtain

$$(A) \int_0^1 f\, dW = \delta(f) + \int_0^1 D_t f(t-)\, dt$$

with Malliavin derivative D_t. In general, this integral differs from the Stratonovitch integral, but we also get

$$(A)\int_0^1 cf\, dW = c\,(A)\int_0^1 f\, dW$$

for random constants c.

1. Fractional integrals and derivatives

Let $\mathfrak{L}^n$ be Lebesgue measure in $\mathbb{R}^n$. Integration with respect to $\mathfrak{L}(dx)$ will be denoted by dx. For $p \geq 1$ let $L_p = L_p(a,b)$ be the space of complex-valued functions on $\mathbb{R}$ such that $||f||_{L_p} = \left(\int_a^b |f(x)|^p\, dx\right)^{1/p} < \infty$. (Similarly for $p = \infty$.) Sometimes the values of f in a neighborhood of $[a,b]$ are of interest. Functions which agree at Lebesgue-almost all points are usually identified.

An exhaustive survey on classical fractional calculus may be found in Samko, Kilbas and Marichev [11]. We recall some important notions and results presented there.

For $f \in L_1$ and $\alpha > 0$ the *left*- and *right-sided fractional Riemann–Liouville integrals* of f *of order* α on (a,b) is given at almost all x by

$$I^\alpha_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y)\, dy \tag{7}$$

$$I^\alpha_{b-}f(x) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)}\int_x^b (y-x)^{\alpha-1} f(y)\, dy\ , \tag{8}$$

respectively, where Γ denotes the Euler function.

They extend the usual n-th order iterated integrals of f for $\alpha = n \in \mathbb{N}$. We have the *first composition formula*

$$I^\alpha_{\substack{a+\\(b-)}}\left(I^\beta_{\substack{a+\\(b-)}} f\right) = I^{\alpha+\beta}_{\substack{a+\\(b-)}} f\ . \tag{9}$$

If $f \in L_p,\ g \in L_q,\ p \geq 1,\ q \geq 1,\ 1/p + 1/q \leq 1 + \alpha$, where $p > 1$ and $q > 1$ for $1/p + 1/q = 1 + \alpha$, then the *first integration-by-parts rule* holds:

$$\int_a^b f(x) I^\alpha_{a+} g(x)\, dx = (-1)^\alpha \int_a^b g(x) I^\alpha_{b-} f(x)\, dx\ . \tag{10}$$

Fractional differentiation may be introduced as an inverse operation. For our purposes it is sufficient to work with a class of functions where this inversion is well-determined and the Riemann–Liouville derivatives agree with the (more general) version in the sense of Weyl and Marchaud:

For $p \geq 1$ let $I^{\alpha}_{\substack{a+\\(b-)}}(L_p)$ be the class of functions f which may be represented as an $I^{\alpha}_{\substack{a+\\(b-)}}$-integral of some L_p-function φ. If $p > 1$ this property is equivalent to $f \in L_p$ and L_p-convergence of the integrals

$$\int_a^{x-\epsilon} \frac{f(x) - f(y)}{(x-y)^{\alpha+1}}\, dy \quad \left(\int_{x+\epsilon}^{b} \frac{f(x) - f(y)}{(y-x)^{\alpha+1}}\, dy \right)$$

as function in $x \in (a,b)$ as $\epsilon \searrow 0$ putting $f(y) = 0$ if $x \notin [a,b]$ (cf. [11], §13). Moreover $\alpha - 1/p$, for $\alpha p < 1$ one knows that $I^{\alpha}_{a+}(L_p) = I^{\alpha}_{b-}(L_p) \subset L_q$ with $1/q = 1/p - \alpha$. If $\alpha p > 1$ any $f \in I^{\alpha}_{\substack{a+\\(b-)}}(L_p)$ is Hölder continuous of order $\alpha - 1/p$ on the interval (a,b).

It can be shown that the function φ in the above representation $f = I^{\alpha}_{\substack{a+\\(b-)}}\varphi$ is unique in L_p on (a,b) and for $0 < \alpha < 1$ it agrees a.e. with the *left*-(*right*-)*sided Riemann–Liouville derivative* of f *of order* α

$$D^{\alpha}_{a+} f(x) := \ 1_{(a,b)}(x) \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^{\alpha}}\, dy \tag{11}$$

$$\left(D^{\alpha}_{b-} f(x) := 1_{(a,b)}(x) \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y-x)^{\alpha}}\, dy \right) . \tag{12}$$

The corresponding *Weyl representation* reads

$$D^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}}\, dy \right) 1_{(a,b)}(x) \tag{13}$$

$$\left(D^{\alpha}_{b-} f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}}\, dy \right) 1_{(a,b)}(x) \right) \tag{14}$$

where the convergence of the integrals at the singularity $y = x$ holds pointwise for almost all x if $p = 1$ and in the L_p-sense if $p > 1$. (The more familiar infinite versions of the Weyl derivatives are given by

$$D^{\alpha}_{+} f(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x-y)}{y^{\alpha+1}}\, dy \tag{13'}$$

$$D^{\alpha}_{-} f(x) := \frac{\alpha(-1)^{\alpha}}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x+y)}{y^{\alpha+1}}\, dy \ . \tag{14'}$$

Since f vanishes outside $[a,b]$ we obtain

$$D^{\alpha}_{\substack{a+\\(b-)}} f(x) = 1_{(a,b)}(x) D^{\alpha}_{\substack{+\\(-)}} f(x) \ .$$

Note that in the literature the factor $(-1)^{\alpha}$ is usually omitted, though it was originally used by Liouville. It appears appropriate for our integral construction and plays also a role in fractional Fourier transformations (c.f. [14]).

Recall that by construction for $f \in I^{\alpha}_{\substack{a+\\(b-)}}(L_p)$,

$$I^{\alpha}_{\substack{a+\\(b-)}}(D^{\alpha}_{\substack{a+\\(b-)}} f) = f \quad . \tag{15}$$

We also have

$$D^{\alpha}_{\substack{a+\\(b-)}}(I^{\alpha}_{\substack{a+\\(b-)}} f) = f \tag{16}$$

which is valid for general $f \in L_1$.

Straightforward calculation shows that for f continuously differentiable in a neighborhood of $x \in (a, b)$,

$$\lim_{\alpha \to 1} D^{\alpha}_{\substack{a+\\(b-)}} f(x) = f'(x). \tag{17}$$

Here the relationship

$$\lim_{\epsilon \to 0} I^{\epsilon}_{\substack{a+\\(b-)}} h(x) = h(x\underset{(+)}{-}) \tag{18}$$

is used which holds for arbitrary $h \in L_1$ at all points $x \in (a, b)$ where the left (right) limit, $h(x-)(h(x+))$ exists, i.e., at Lebesgue-almost all x. If $h \in L_p$, $p \geq 1$, we can take in (18) the L_p-limit, too. In particular, in (17) L_p-convergence holds for all $f \in L_p$ which are differentiable in the L_p-sense.

Furthermore, (11) and (12) imply

$$\lim_{\alpha \to 0} D^{\alpha}_{\substack{a+\\(b-)}} f(x) = f(x) \tag{19}$$

which is also true in the L_p-sense if $p > 1$. For completeness denote

$$D^{0}_{\substack{a+\\(b-)}} f(x) = f(x), \quad I^{0}_{\substack{a+\\(b-)}}(L_p) = L_p, \text{ and } D^{1}_{\substack{a+\\(b-)}} f(x) = f'(x)$$

if the latter derivative exists.

The following two formulas play an essential role for our integration concept:

$$D^{\alpha}_{\substack{a+\\(b-)}}(D^{\beta}_{\substack{a+\\(b-)}} f) = D^{\alpha+\beta}_{\substack{a+\\(b-)}} f \tag{20}$$

if $f \in I^{\alpha+\beta}_{\substack{a+\\(b-)}}(L_1)$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \leq 1$ (*second composition formula*),

$$(-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f(x)\, g(x)\, dx = \int_a^b f(x) D^{\alpha}_{b-} g(x)\, dx \tag{21}$$

provided that $0 \leq \alpha \leq 1$, $f \in I^{\alpha}_{a+}(L_p)$, $g \in I^{\alpha}_{b-}(L_q)$, $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1 + \alpha$ (*second integration-by-parts rule*).

2. An extension of Stieltjes integrals

The calculation rules (3) and (4) for Lebesgue–Stieltjes integrals with respect to smooth functions, the composition formula (20) and the integration-by-part rule (21) suggest the following notion. (In order to avoid the restrictive condition $f(a+) = 0$ or $g(b-) = 0$ at some places we introduce the auxiliary functions f_{a+} and g_{b-} as in (5) and (6) assuming that the right- and left-sided limits always exist when they appear in the formulae.)

Definition. *The (fractional) integral of f with respect to g is defined by*

$$\int_a^b f(x)\,dg(x) = (-1)^\alpha \int_a^b D^\alpha_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x)\,dx + f(a+)(g(b-) - g(a+)) \tag{22}$$

provided that $f_{a+} \in I^\alpha_{a+}(L_p)$, $g_{b-} \in I^{1-\alpha}_{b-}(L_q)$ *for some* $1/p + 1/q \leq 1$, $0 \leq \alpha \leq 1$.

2.1. Proposition *The definition is correct, i.e. independent of the choice of* α.

Proof. If the conditions are fulfilled for (α, p, q) and (α', p', q') with $\alpha' = \alpha + \beta > \alpha$ then we get

$$\begin{aligned}
&(-1)^{\alpha'} \int_a^b D^{\alpha'}_{a+} f_{a+}(x) D^{1-\alpha'}_{b-} g_{b-}(x)\,dx \\
\overset{(20)}{=}\ &(-1)^\alpha (-1)^\beta \int_a^b D^\beta_{a+}(D^\alpha_{a+} f_{a+})(x) D^{1-(\alpha+\beta)}_{b-} g_{b-}(x)\,dx \\
\overset{(21)}{=}\ &(-1)^\alpha \int_a^b D^\alpha_{a+} f_{a+}(x) D^\beta_{b-}(D^{1-(\alpha+\beta)}_{b-} g_{b-})(x)\,dx \\
\overset{(20)}{=}\ &(-1)^\alpha \int_a^b D^\alpha_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x)\,dx\ .
\end{aligned}$$

In order to check the conditions of (21) use (16) and (9). □

Remark. For $\alpha p < 1$ we have $f_{a+} \in I^\alpha_{a+}(L_p)$ iff $f \in I^\alpha_{a+}(L_p)$ and $f(a+)$ exists. In this case the derivatives satisfy the relation

$$\begin{aligned}
D^\alpha_{a+} f_{a+}(x) &= D^\alpha_{a+}(f - f(a+) 1_{(a,b)})(x) \\
&= D^\alpha_{a+} f(x) - \frac{1}{\Gamma(1-\alpha)} \frac{f(a+)}{(x-a)^\alpha}\ 1_{(a,b)}(x)
\end{aligned}$$

(cf. Proposition 2.2 below) and (22) may be rewritten as

$$\int_a^b f(x)\,dg(x) = (-1)^\alpha \int_a^b D^\alpha_{a+} f(x) D^{1-\alpha}_{b-} g_{b-}(x)\,dx\ . \tag{22'}$$

which is determined for general $f \in I^{\alpha}_{a+}(L_p)$ bounded in $a+$. For $\alpha = 0$ and $\alpha = 1$ the integral (22) may be transformed into

$$\int_a^b f(x)\, dg(x) = \int_a^b f(x) g'(x)\, dx \tag{23}$$

and

$$\int_a^b f(x)\, dg(x) = -\int_a^b f'(x) g(x)\, dx + f(b-)g(b-) - f(a+)g(a+) \tag{24}$$

which agrees with the corresponding Lebesgue–Stieltjes integrals (3) and (4), respectively.

Our next aim is to show that for functions g as above with finite variation the integral (22) agrees with the Lebesgue–Stieltjes integral of the functions f under consideration. First we let f be the indicator function of a subinterval $(c, d] \subset (a, b)$.

2.2. Proposition. *If g is Hölder continuous on (a, b) of some order then we have*

(i) $\int_a^b 1_{(c,d]}(x)\, dg(x) = g(d) - g(c)$
(ii) $\int_a^b 1_{(c,b)}(x)\, dg(x) = g(b-) - g(c)$.

Note that on the right hand side $g(c)$ has to be replaced by $g(a+)$ if $c = a$.

Proof. The fractional derivatives of the function $1_{(c,d]}(x)$ may be calculated by means of (13′):

$$\begin{aligned} D^{\alpha}_{a+} 1_{(c,d]}(x) &= 1_{(a,b)}(x) \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{1_{(c,d]}(x) - 1_{(c,d]}(x-y)}{y^{\alpha+1}}\, dy \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \left[1_{(c,d]}(x) \int_0^{\infty} \frac{1 - 1_{(c,d]}(x-y)}{y^{\alpha+1}}\, dy \right. \\ &\qquad \left. - 1_{(a,b)\setminus(c,d]}(x) \int_0^{\infty} \frac{1_{(c,d]}(x-y)}{y^{\alpha+1}}\, dy \right] \\ &= \frac{1}{\Gamma(1-\alpha)} \left[1_{(c,d]}(x) \alpha \int_{x-c}^{\infty} \frac{1}{y^{\alpha+1}}\, dy \right. \\ &\qquad \left. - 1_{(d,b)}(x) \int_{x-d}^{x-c} \frac{1}{y^{\alpha+1}}\, dy \right] . \end{aligned}$$

Thus,

$$D_{a+}^{\alpha}\ 1_{(c,d]}(x) = \frac{1}{\Gamma(1-\alpha)}\left[1_{(c,b)}(x)\ \frac{1}{(x-c)^{\alpha}} - 1_{(d,b)}(x)\ \frac{1}{(x-d)^{\alpha}}\right]\ . \tag{25}$$

Similarly,

$$D_{a+}^{\alpha}\ 1_{(c,b)}(x) = \frac{1}{\Gamma(1-\alpha)}\ 1_{(c,b)}(x)\frac{1}{(x-c)^{\alpha}}\ . \tag{26}$$

Taking the I_{a+}^{α}-integral of the right-hand side we can see that $1_{(c,d]} \in I_{a+}^{\alpha}(L_p)$ iff $\alpha p < 1$. Further, if λ is the Hölder exponent of the function g then g_{b-} lies in $I_{b-}^{\epsilon}(L_q)$ for any q and $\epsilon < \lambda$. Hence, the conditions of (22) are fulfilled for arbitrary $\alpha = 1 - \epsilon$, $\epsilon < \lambda$, i.e.,

$$\begin{aligned}\int_a^b 1_{(c,d]}(x)\ dg(x) &= (-1)^{1-\epsilon}\int_a^b\ D_{a+}^{1-\epsilon}1_{(c,d]}(x)\ D_{b-}^{\epsilon}g_{b-}(x)\ dx\\ &= (-1)^{1-\epsilon}\frac{1}{\Gamma(\epsilon)}\left[\int_0^{b-c} x^{\epsilon-1}D_{b-}^{\epsilon}g_{b-}(c+x)\ dx\right.\\ &\quad\left. - \int_0^{b-d} x^{\epsilon-1}D_{b-}^{\epsilon}g_{b-}(d+x)\ dx\right]\\ &= (-1)^{1-\epsilon}(-1)^{\epsilon}\left[I_{b-}^{\epsilon}(D_{b-}^{\epsilon}g_{b-})(c) - I_{b-}^{\epsilon}(D_{b-}^{\epsilon}g_{b-})(d)\right]\\ &= -[g(c) - g(d)]\end{aligned}$$

according to (15).

The arguments for (ii) are similar. □

By linearity this result extends to step functions: Let $\mathscr{P} = \{a = x_0 < x_1 < \cdots < x_n < b\}$ be any partition of (a,b) and $f_{\mathscr{P}} := \sum_{i=0}^{n-1} f_i 1_{(x_i,x_{i+1}]} + f_n 1_{(x_n,b)}$ for some complex values f_i.

2.3. Corollary. *If g is Hölder continuous on (a,b) we have*

$$\int_a^b f_{\mathscr{P}}(x)\ dg(x) = \sum_{i=0}^{n-1} f_i\ (g(x_{i+1}) - g(x_i)) + f_n\ (g(b-) - g(x_n))\ .$$

We now turn to comparison with the Lebesgue-Stieltjes integral under the condition that g has bounded variation. In the complex case the Lebesgue–Stieltjes integral considered in the introduction may be understood in the real vector-valued sense via coordinate representation.

2.4. Theorem. *Suppose that g has bounded variation with variation measure μ and f and g satisfy the conditions of* (22).

(i) *If* $\int_a^b I_{a+}^{\alpha}(|D_{a+}^{\alpha} f_{a+}|)(x)\mu(dx) < \infty$ *then we have* $\int_a^b f(x)\, dg(x) = \text{(L-S)} \int_a^b f(x)\, dg(x)$.
(ii) *The integrals also agree if f is bounded and right-(or left-)continuous at μ–a.a. points.*

Remark. As a special case of (ii) we obtain for any continuous f as above

$$\int_a^b f(x)\, dg(x) = \text{(R-S)} \int_a^b f(x)\, dg(x) \ .$$

Proof of the theorem. The condition of (i) and Fubini justify the changes of the order of integration in the following Lebesgue–Stieltjes integrals. By (15) we get

$$\begin{aligned}\text{(L-S)} \int_a^b f\, dg = \text{(L-S)} \int_a^b & I_{a+}^{\alpha}(D_{a+}^{\alpha} f_{a+})(x)\, dg(x) \\ & + f(a+)(g(b-) - g(a+)) \ .\end{aligned}$$

The integral on the right-hand side equals

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)} \int_a^b \int_a^x (x-y)^{\alpha-1} D_{a+}^{\alpha} f_{a+}(y)\, dy\, dg(x) \\ &\quad = \frac{1}{\Gamma(\alpha)} \int_a^b D_{a+}^{\alpha} f_{a+}(y) \int_y^b (x-y)^{\alpha-1}\, dg(x)\, dy \\ &\quad = \int_a^b D_{a+}^{\alpha} f_{a+}(y) \left[\frac{1-\alpha}{\Gamma(\alpha)} \int_y^b \int_x^{\infty} (z-y)^{\alpha-2}\, dz\, dg(x) \right] dy \ .\end{aligned}$$

The expression in the brackets may be rewritten by

$$\begin{aligned}&\frac{1-\alpha}{\Gamma(\alpha)} \int_y^{\infty} \int_y^{\min(z,b)} dg(x)\ (z-y)^{\alpha-1}\, dz \\ &\quad = -\frac{1-\alpha}{\Gamma(\alpha)} \int_y^{\infty} \frac{g(y) - g(\min(z,b))}{(z-y)^{1-\alpha+1}}\, dz \\ &\quad = -\frac{1-\alpha}{\Gamma(\alpha)} \int_y^b \frac{g(y) - g(z)}{(z-y)^{1-\alpha+1}}\, dz - \frac{1}{\Gamma(\alpha)} \frac{g(y) - g(b-)}{(b-y)^{1-\alpha}} \\ &\quad = (-1)^{\alpha} D_{b-}^{1-\alpha} g_{b-}(y)\end{aligned}$$

according to (14). Substituting this under the above integral we infer that the primary integral equals $(-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f_{a+}(y) D_{b-}^{1-\alpha} g_{b^-}(y)\, dy + f(a+)(g(b-) - g(a+))$, i.e. (i) is proved.

In order to show (ii) note that both integrals agree with $-\int_a^b f'(x) g(x)\, dx + f(b-)g(b-) - f(a+)g(a+)$ if f is smooth. We will approximate general f by smooth functions so that both types of integrals converge to those of f. Let (α, p) satisfy the conditions of (22). Then $D^{\alpha}_{a+} f_{a+}$ is an L_p-function.

Let k (or k^-) be a nonnegative smooth function vanishing outside $[0,1]$ (or $[-1,0]$) such that $\int_0^1 k(x)\,dx = 1$ (or $\int_{-1}^0 k^-(x)\,dx = 1$). By

$$k_N^{(-)}(x) := N\ k^{(-)}(Nx) \tag{27}$$

we get a standard familiy of smoothing kernels converging to the δ-function as $N \to \infty$.

For the convolution $f_N := f_{a+} * k_N$ we obtain

$$D^{\alpha}_{a+} f_N = 1_{(a,b)}(D^{\alpha}_{a+} f_{a+}) * k_N\ \ . \tag{28}$$

The right-hand side converges in L_p to $D^{\alpha}_{a+} f_{a+}$ as $N \to \infty$. Further, $f_N(a+) = 0$. Hence, by the Hölder inequality we obtain

$$\begin{aligned}
&(-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x)\,dx \\
&= \lim_{N\to\infty} (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f_N(x) D^{1-\alpha}_{b-} g_{b-}(x)\,dx \\
&= \lim_{N\to\infty} (- \int_a^b f'_N(x) g_{b-}(x)\,dx) \\
&= \lim_{N\to\infty} (\text{L-S}) \int_a^b f_N(x)\,dg_{b-}(x) \\
&= \lim_{N\to\infty} (\text{L-S}) \int_a^b f_N(x)\,dg(x)\ \ .
\end{aligned}$$

The right-sided continuity of f at x yields

$$\lim_{N\to\infty} f_{a+} * k_N(x) = f_{a+}(x)\ \ .$$

Therefore Lebesgue's bounded convergence theorem implies

$$\begin{aligned}
&\lim_{N\to\infty} (\text{L-S}) \int_a^b f_{a+} * k_N(x)\,dg(x) \\
&= (\text{L-S}) \int_a^b f_{a+}(x)\,dg(x) \\
&= (\text{L-S}) \int_a^b f(x)\,dg(x) - f(a+)(g(b-) - g(a+))
\end{aligned}$$

which leads to the assertion.

The case of left-sided continuity is similar. Here it is appropriate to use the kernel k^-. □

Remark. It turns out that in (ii) the Lebesgue–Stieltjes integral does not depend on the choice of right- or left-sided limits of f. This comes from the conditions of (22). In case of discontinuous f they force a certain Hölder continuity of g.

At the end of this section we will show that our integral (22) is an *additive function of the boundary*. Let $a \leq x < y < z \leq b$.

2.5. Theorem.

(i) $\int_x^y f\,dg = \int_a^b 1_{(x,y)} f\,dg$
if for both the integrals the conditions of definition (22) *are fulfilled.*
(ii) $\int_x^y f\,dg + \int_y^z f dg = \int_x^z f dg - f(y)(g(y+) - g(y-))$
if all summands are determined as in (22).

Proof. Let k_N^- be the family of smoothing kernels introduced in (27). We first will approximate the function g_{b-} by the smooth functions $g_N := g_{b-} * k_N^-$ so that

$$D_{b-}^{1-\alpha} g_N = 1_{(a,b)} (D_{b-}^{1-\alpha} g_{b-}) * k_N^- \tag{29}$$

and $g_N(b-) = 0$. Then we obtain by L^q convergence for $x > a$ (the case $x = a$ is similar)

$$\begin{aligned}
\int_a^b 1_{(x,y)} f\,dg &= (-1)^\alpha \int_a^b D_{a+}^\alpha 1_{(x,y)} f(u) D_{b-}^{1-\alpha} g_{b-}(u)\,du \\
&= \lim_{N\to\infty} (-1)^\alpha \int_a^b D_{a+}^\alpha 1_{(x,y)} f(u) D_{b-}^{1-\alpha} g_N(u)\,du \\
&= \lim_{N\to\infty} \int_a^b 1_{(x,y)}(u) f(u) g_N'(u)\,du \\
&= \lim_{N\to\infty} \int_x^y f(u)\; g_{b-} * (k_N^-)'(u)\,du \\
&= \lim_{N\to\infty} \int_x^y f(u)\; g_{y-} * (k_N^-)'(u)\,du \ .
\end{aligned}$$

The last equality follows from the asymptotic equivalence of the functions $g_{b-} * (k_N^-)'$ and $g_{y-} * (k_N^-)'$ on the interval (x, y). Further, the conditions of (22) are also fulfilled for the interval (x, y) for some (α', p', q'). Therefore we may continue the above equations by

$$\begin{aligned}
&\lim_{N\to\infty} \int_x^y D_{x+}^{\alpha'} f_{x+}(u)\; D_{y-}^{1-\alpha'} g_{y-} * k_N^-(u)\,du + f(x+)(g(y-) - g(x+)) \\
&\quad = \int_x^y D_{x+}^{\alpha'} f_{x+}(u) D_{y-}^{1-\alpha'} g_{y-}(u)\,du + f(x+)(g(y-) - g(x+)) = \int_x^y f\,dg \ .
\end{aligned}$$

Thus (i) is proved.
For (ii) we use similar arguments in order to get

$$\int_x^y f\,dg + \int_y^z f\,dg - f(x+)(g(y-) - g(x+)) - f(y+)(g(z-) - g(y+))$$

$$= \lim_{N\to\infty}\left(\int_x^y f_{x+}(u)\; g*(k_N^-)'(u)\,du + \int_y^z f_{y+}(u)\; g*(k_N^-)'(u)\,du\right)$$

$$= \lim_{N\to\infty}\int_x^z f_{x+}(u)\; g*(k_N^-)'(u)\,du - \left(f(x+) - f(y+)\right)\int_y^z g*k_N^-(u)\,du\right)$$

$$= \int_x^z f\,dg - f(x+)(g(z-) - g(x+)) - (f(x+) - f(y+))(g(z-) - g(y-))\,.$$

□

Remark. For $\alpha p < 1$, $f \in I^{\alpha}_{a+}(L_p)$ being bounded in $x+$ and $y+$, $g_{b-} \in I^{1-\alpha}_{b-}$ (where g is continuous), $g(a+)$ existing, $\frac{1}{p} + \frac{1}{q} \le 1$ we get similarly

$$\int_x^y f\,dg + \int_y^z f\,dg = \int_x^z f\,dg$$

by means of (22′).

3. Backward integrals and integration by parts

The construction (22), i.e.,

$$\int_a^b f\,dg = (-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f_{a+}(x)D^{1-\alpha}_{b-}g_{b-}(x)\,dx + f(a+)(g(b-) - g(a+))$$

is directed because of the choice of left-sided derivatives of f and right-sided derivatives of g. We will also call this expression the *forward integral* of f with respect to g. Similarly, we may introduce the *backward integral*

$$\int_a^b dg(x)\,f(x) := (-1)^{-\alpha'}\int_a^b D^{\alpha'}_{b-}f_{b-}(x)D^{1-\alpha'}_{a+}g_{a+}(x)\,dx + f(b-)(g(b-) - g(a+)) \qquad (30)$$

if $f_{b-} \in I^{\alpha'}_{b-}(L_{p'})$, $g_{a+} \in I^{1-\alpha'}_{a+}(L_{q'})$ for some $1/p' + 1/q' \le 1$, $0 \le \alpha' \le 1$. Then the backward versions of (22′)–(26) may be proved by completely analogous arguments. In particular, for indicator functions f or smooth functions f or g the forward and backward integrals agree. Generally, the following holds.

3.1. Theorem. *If f and g satisfy the conditions of* (22) *and* (30) *then we have*

(i) $\int_a^b f\ dg = \int_a^b\ dg\ f$.
(ii) $\int_a^b f\ dg = f(b-)g(b-) - f(a+)g(a+) - \int_a^b g\ df$

(*integration-by-part formula*).

Proof. Using the approximations (28) and (29) for the left- and right-sided derivatives in the forward, as well as the backward integrals we infer from convergences in L_p, L_q and $L_{p'}$, $L_{q'}$, respectively, $\int_a^b f\ dg = \lim_{N\to\infty} \int_a^b f * k_N(x)\ g * (k_N^-)'(x)\ dx = \int_a^b\ dg\ f$, i.e., (i). (ii) is a consequence, since by definition,

$$\begin{aligned}\int_a^b f\ dg &= -\int_a^b\ df\ g + f(a+)(g(b-) - g(a+)) \\ &\quad + g(b-)(f(b-) - f(a+)) \\ &= -\int_a^b g\ df + f(b-)g(b-) + f(b-)g(b-) - f(a+)g(a+)\ .\end{aligned}$$

Remark. Let $H^\lambda = H^\lambda(a,b)$ be the space of functions being Hölder continuous of order λ on the interval (a,b). Then the conditions of Theorem 3.1 are fulfilled if $f \in H^\lambda$, $g \in H^\mu$, $\lambda + \mu > 1$. (In this case we may choose $p = q = \infty$ for $\alpha < \lambda$, $1 - \alpha < \mu$.) In the next section we will study this situation in more detail.

4. The case of Hölder continuous functions

4.1 Approximation by step functions

For arbitrary partitions $\mathscr{P}_\Delta$ as before any Hölder continuous function f on (a,b) may be approximated by the special step functions

$$\widetilde{f}_{\mathscr{P}_\Delta} := \sum_{i=0}^n f(x_i) 1_{(x_i, x_{i+1}]}$$

in the following sense.

4.1.1. Theorem. *If* $f \in H^\lambda$ *for some* $0 < \lambda \le 1$ *then we have*

(i) $\lim_{\Delta\to 0}\ \sup_{\mathscr{P}_\Delta} \|\widetilde{f}_{\mathscr{P}_\Delta} - f\|_{L_\infty(a,b)} = 0$
(ii) $\lim_{\Delta\to 0}\ \sup_{\mathscr{P}_\Delta} \|D^\alpha_{\substack{a+\\(b-)}} (f_{\mathscr{P}_\Delta})_{\substack{a+\\(b-)}} - D^\alpha_{\substack{a+\\(b-)}} f_{\substack{a+\\(b-)}}\|_{L_1(a,b)} = 0$

for any $\alpha < \lambda$.

Proof. (i) is obvious.

For (ii) we will prove only the left-sided version. (The right-sided case is analogous.) Let $H(\lambda)$ be the Hölder constant of f. By definition,

$$\Gamma(1-\alpha)|D^{\alpha}_{a+}(\widetilde{f}_{\mathscr{P}_\Delta})_{a+}(x) - D^{\alpha}_{a+}f_{a+}(x)|$$
$$= \left| \frac{\widetilde{f}_{\mathscr{P}_\Delta}(x) - f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{\widetilde{f}_{\mathscr{P}_\Delta}(x) - f(x) - (\widetilde{f}_{\mathscr{P}_\Delta}(y) - f(y))}{(x-y)^{\alpha+1}}\, dy \right| .$$

The L_1-norm of the first summand of the last sum may be estimated by $H(\lambda)(b-a)^{1-\alpha}\Delta^{\lambda}$.

For $x \in (x_i, x_{i+1}]$ the second summand, say $S_{\mathscr{P}_\Delta}(x)$, may be splitted into

$$\alpha \sum_{k=0}^{i-1} \int_{x_k}^{x_{k+1}} \frac{\widetilde{f}_{\mathscr{P}_\Delta}(x) - f(x) - (\widetilde{f}_{\mathscr{P}_\Delta}(y) - f(y))}{(x-y)^{\alpha+1}}\, dy$$
$$+ \alpha \int_{x_i}^{x} \frac{\widetilde{f}_{\mathscr{P}_\Delta}(x) - f(x) - (\widetilde{f}_{\mathscr{P}_\Delta}(y) - f(y))}{(x-y)^{\alpha+1}}\, dy$$
$$= \alpha \sum_{k=0}^{i-1} \int_{x_k}^{x_{k+1}} \frac{f(x_i) - f(x) - (f(x_k) - f(y))}{(x-y)^{\alpha+1}}\, dy$$
$$+ \alpha \int_{x_i}^{x} \frac{f(x_i) - f(x) - (f(x_i) - f(y))}{(x-y)^{\alpha+1}}\, dy \ .$$

Therefore the Hölder continuity of f leads here to the estimation

$$|S_{\mathscr{P}_\Delta}(x)| \le H(\lambda) \sum_{i=0}^{n} 1_{(x_i,x_{i+1}]}(x) \Bigg[(x-x_i)^{\lambda} \alpha \int_a^{x_i} \frac{1}{(x-y)^{\alpha+1}}\, dy$$
$$+ \alpha \sum_{k=0}^{i-1} (x_{k+1}-x_k)^{\lambda} \int_{x_k}^{x_{k+1}} \frac{1}{(x-y)^{\alpha+1}}\, dy$$
$$+ \alpha \int_{x_i}^{x} \frac{(x-y)^{\lambda}}{(x-y)^{\alpha+1}}\, dy \Bigg]$$
$$\le H(\lambda) \sum_{i=0}^{n} 1_{(x_i,x_{i+1}]}(x) \Bigg[(x-x_i)^{\lambda-\alpha} + \alpha \sum_{k=0}^{i-1} (x_{k+1}-x_k)^{\lambda}$$
$$\times \int_{x_k}^{x_{k+1}} \frac{1}{(x-y)^{\alpha+1}}\, dy + \frac{\alpha}{\lambda-\alpha}\ (x-x_i)^{\lambda-\alpha} \Bigg] \ .$$

Hence,

$$\begin{aligned}
\|S_{\mathscr{P}_\Delta}\|_{L_1} &\le H(\lambda)\Bigg[\frac{\lambda}{\lambda-\alpha}\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}}(x-x_i)^{\lambda-\alpha}\,dx \\
&\quad + \alpha\sum_{i=0}^{n}\sum_{k=0}^{i-1}(x_{k+1}-x_k)^{\lambda}\int_{x_i}^{x_{i+1}}\int_{x_k}^{x_{k+1}}\frac{1}{(x-y)^{\alpha+1}}\,dy\,dx\Bigg] \\
&= H(\lambda)\Bigg[\frac{\lambda}{\lambda-\alpha}\,\frac{1}{\lambda-\alpha+1}\sum_{i=0}^{n}(x_{i+1}-x_i)^{\lambda-\alpha+1} \\
&\quad + \sum_{k=0}^{n-1}(x_{k+1}-x_k)^{\lambda}\int_{x_k}^{x_{k+1}}\int_{x_{k+1}}^{b}\frac{1}{(x-y)^{\alpha+1}}\,dx\,dy\Bigg] \\
&\le H(\lambda)\left[\frac{\lambda}{\lambda-\alpha}\,\frac{1}{\lambda-\alpha+1}+\frac{1}{1-\alpha}\right]\sum_{i=0}^{n}(x_{i+1}-x_i)^{\lambda-\alpha+1} \\
&\le H(\lambda)\left[\frac{\lambda}{\lambda-\alpha}\,\frac{1}{\lambda-\alpha+1}+\frac{1}{1-\alpha}\right](b-a)\Delta^{\lambda-\alpha}
\end{aligned}$$

which completes the proof of (ii). □

4.2 Interpretation as Riemann–Stieltjes integral

4.2.1. Theorem. *If $f \in H^{\lambda}$, $g \in H^{\mu}$ for some $\lambda+\mu>1$ the Riemann–Stieltjes integral* (R-S) $\int_a^b f\,dg$ *in the sense of* (2) *exists and agrees with the forward and backward integrals* $\int_a^b f\,dg$ *and* $\int_a^b dg\,f$ *in the sense of* (22) *and* (30).

Proof. Let $f_{\mathscr{P}_\Delta}$ and $\widetilde{f}_{\mathscr{P}_\Delta}$ be the step functions used in (2) and Theorem 4.1.1, respectively. We estimate the difference of their Riemann–Stieltjes sums by

$$\begin{aligned}
&\sup_{\mathscr{P}_\Delta}\left|\sum_{i=0}^{n}f(x_i^*)(g(x_{i+1})-g(x_i))-\sum_{i=0}^{n}f(x_i)(g(x_{i+1})-g(x_i))\right| \\
&\le \sup_{\mathscr{P}_\Delta}\sum_{i=0}^{n}|f(x_i^*)-f(x_i)||g(x_{i+1})-g(x_i)| \\
&\le H(\lambda)H(\mu)\sup_{\mathscr{P}_\Delta}\sum_{i=0}^{n}(x_{i+1}-x_i)^{\lambda+\mu} \\
&\le H(\lambda)H(\mu)(b-a)\Delta^{\lambda+\mu-1}
\end{aligned}$$

where $H(\lambda)$ and $H(\mu)$ are the Hölder constants of f and g, respectively. Therefore it is enough to prove the convergence of the Riemann–Stieltjes sums $\sum_{i=0}^{n} f(x_i)(g(x_{i+1})-g(x_i))$ to $\int_a^b f\,dg$ which agrees with $\int_a^b dg\, f$ by Theorem 3.1 (i). According to corollary 2.3 these sums may be interpreted as the forward integrals

$$\int_a^b \widetilde{f}_{\mathscr{P}_\Delta}\, dg = (-1)^\alpha \int_a^b D^\alpha_{a+}(\widetilde{f}_{\mathscr{P}_\Delta})_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x)\, dx$$
$$+ f(a+)(g(b-) - g(a+))$$

for any $1 - \mu < \alpha < \lambda$. By Theorem 4.1.1 (i) the right-hand side tends to

$$(-1)^\alpha \int_a^b D^\alpha_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x)\, dx + f(a+)(g(b-) - g(a+)) = \int_a^b f\, dg$$

as $\Delta \to 0$ uniformly in the partitions $\mathscr{P}_\Delta$ since $D^{1-\alpha}_{b-} g_{b-}$ is bounded. □

4.3 A change-of-variable formula

It is well-known that the chain rule

$$dF(f(x)) = F'(f(x))\, df(x)$$

of classical real differentiation theory does not hold for functions f of Hölder exponent 1/2 arising as sample paths of stochastic processes which are semimartingales (cf. section 5). However, it follows from Theorem 4.2.1 that for functions of Hölder exponent greater than 1/2 the classical formula remains valid in the sense of Riemann–Stieltjes integration:

4.3.1. Theorem. *Let $f \in H^\lambda(a,b)$ and $F \in C^1(\mathbb{R})$ be real-valued functions such that $F' \circ f \in H^\mu(a,b)$ for some $\lambda + \mu > 1$. Then we have for any $y \in (a,b)$*

$$F(f(y)) - F(f(a)) = \int_a^y F'(f(x))\, df(x) \ .$$

Proof. For arbitrary partitions $\mathscr{P}_\Delta$ as above the mean value theorem for F and the continuity of f imply

$$F(f(y)) - F(f(a)) = \sum_{i=0}^n F(f(x_{i+1})) - F(f(x_i))$$
$$= \sum_{i=0}^n F'(f(\widetilde{x}_i))(f(x_{i+1}) - f(x_i))$$

for some $\widetilde{x_i} \in [x_i, x_{i+1}]$. The last expression tends to $\int_a^y F'(f(x)\, df(x)$ as $\Delta \to 0$ by Theorem 4.2.1. □

Remark. The conditions of this theorem are satisfied if $f \in H^\lambda(a,b)$ for some $\lambda > 1/2$ and F is a C^1-function with Lipschitz derivative.

A more general variant of Theorem 4.3.1 for $F \in C^1(\mathbb{R} \times (a,b))$ and $F_1'(f(\cdot),\cdot) \in H^\mu(a,b), \quad \lambda + \mu > 1$, reads

$$F(f(y),y) - F(f(a),a) = \int_a^y F_1'(f(x),x)\, df(x) + \int_a^y F_2'(f(x),x)\, dx \tag{31}$$

where F_1' and F_2' are the partial derivatives of F with respect to the first and second variable, respectively. The proof is left to the reader.

Example. If $f \in H^\lambda(a,b)$ *for some* $\lambda > 1/2$ *we may choose in* 4.3.1 $F(u) = u^2$ *and obtain*

$$\int_a^y f(x)\, df(x) = \frac{1}{2}(f(y)^2 - f(a+)^2) \ . \tag{32}$$

4.4 The integral as function of the boundary

An immediate consequence of the interpretation as Riemann–Stieltjes integral for $f \in H^\lambda$, $g \in H^\mu$, $\lambda + \mu > 1$, is the additive dependence on the boundary which has already been proved in Theorem 2.5 by means of smoothing.

Since $\int_x^y f\, dg = (-1)^\alpha \int_x^y D_{x+}^\alpha f_x(u) D_{y-}^{1-\alpha} g_y(u)\, du + f(x)(g(y) - g(x))$ if $1 - \mu < \alpha < \lambda$, $a < x < y < b$, and the derivatives in the last integral are bounded we may estimate $|\int_x^y f\, dg| \le \text{const}(y - x) + \text{const}(y - x)^\mu$ and obtain that the integral as function of the upper or lower boundary is Hölder continuous of order μ:

4.4.1. Proposition. *Under the above conditions we have*

$$1_{(a,b)} \int_a^{(\cdot)} f\, dg \in H^\mu(a,b) \ \text{ and } 1_{(a,b)} \int_{(\cdot)}^b f\, dg \in H^\mu(a,b) \ .$$

In particular, for $h \in H^\lambda$, $g \in H^\mu$, $\lambda + \mu > 1$, we may consider the integrals

$$\varphi(x) := \int_a^x h(y)\, dg(y)\ 1_{(a,b)}(x)$$

and

$$\psi(x) := -\int_x^b h(y)\, dg(y)\; 1_{(a,b)}(x)$$

as functions from $H^\mu(a,b)$.

4.4.2. Theorem. *Under the above conditions we have*

$$\int_a^b f(x)h(x)\, dg(x) = \int_a^b f(x)\, d\varphi(x) = \int_a^b f(x)\, d\psi(x) \ .$$

Proof. For the step functions $\widetilde{f}_{\mathscr{P}_\Delta}(x) = \sum_{i=0}^n f(x_i)1_{(x_i,x_{i+1}]}(x)$ we get from Corollary 2.3 and additivity of the integral

$$\begin{aligned}\int_a^b \widetilde{f}_{\mathscr{P}_\Delta}(x)\, d\varphi(x) &= \sum_{i=0}^n f(x_i)(\varphi(x_{i+1}) - \varphi(x_i))\\ &= \sum_{i=0}^n f(x_i) \int_{x_i}^{x_{i+1}} h(y)\, dg(y)\\ &= \sum_{i=0}^n \int_{x_i}^{x_{i+1}} \widetilde{f}_{\mathscr{P}_\Delta}(y)h(y)\, dg(y)\\ &= \int_a^b \widetilde{f}_{\mathscr{P}_\Delta}(y)h(y)\, dg(y) \ .\end{aligned}$$

Theorem 4.2.1 implies

$$\lim_{\Delta\to 0} \int_a^b \widetilde{f}_{\mathscr{P}_\Delta}(x)\, d\varphi(x) = \int_a^b f(x)\, d\varphi(x) \ .$$

In order to show

$$\lim_{\Delta\to 0} \int_a^b \widetilde{f}_{\mathscr{P}_\Delta}(y)h(y)\, dg(y) = \int_a^b f(y)h(y)\, dg(y)$$

recall that according to the proof of Theorem 4.2.1

$$\left| \int_a^b \widetilde{f}_{\mathscr{P}_\Delta}(y)h(y)\, dg(y) - \int_a^b \widetilde{f}_{\mathscr{P}_\Delta}(y)\widetilde{h}_{\mathscr{P}_\Delta}(y)\, dg(y) \right| \le \text{const}\ \Delta^{\lambda-\alpha}$$

and

$$\lim_{\Delta\to 0} \int_a^b \widetilde{f}_{\mathscr{P}_\Delta}(y)\widetilde{h}_{\mathscr{P}_\Delta}(y)\, dg(y) = \int_a^b f(y)h(y)\, dg(y) \ .$$

Hence,

$$\int_a^b f(x)\ \varphi(dx) = \int_a^b f(y)h(y)\ dg(y)\ \ .$$

The arguments for ψ instead of φ are similar. □

5. Applications to stochastic calculus

5.1 Integration with respect to fractional Brownian motion

A modern presentation of the theory of stochstic integration with respect to semimartingales may be found in Protter [10] and in Winkler and v. Weizsäcker [13]. These books also contain many references to related literature. Semimartingales provide the most general class of stochastic processes for which a stochastic calculus has been developed. In particular, stochastic differential equations are treated.

An important problem, e.g., in finance mathematics is to determine similar differential equations for fractional Brownian motion as an appropriate noise model for real stock-market processes with dependent increments. The study of *fractional Brownian motion* B^H as a real-valued Gaussian process on $[0, +\infty)$ with stationary increments of mean zero and variance

$$\mathbb{E}(B^H(t+s) - B^H(t))^2 = s^{2H}\ \ ,$$

(where $0 < H < 1$) goes back to Kolmogorov and Jaglom (cf. the references in [5]). A representation in terms of a Fourier transform of ordinary Brownian motion $B = B^{1/2}$ was given in Hunt [2]. The name of the process was created in Mandelbrot and van Ness [5] who called the parameter H indicating a certain scaling self-similarity the *Hurst coefficient* of the motion. For more details see Kahane [4].

One can show that B^H has a version with sample paths of Hölder exponent H, i.e. of Hölder continuity of all orders $\lambda < H$ on any finite interval $[0, T]$ with probability 1. The quadratic variation on $[a, b]$ equals

$$\lim_{\Delta\to 0} \sum_i (B^H(t_{i+1}) - B^H(t_i))^2 = \begin{cases} \infty & \text{if } H < 1/2 \\ b - a & \text{if } H = 1/2 \\ 0 & \text{if } H > 1/2 \end{cases}$$

where convergence holds uniformly in $\mathscr{P}_\Delta$ with probability 1 if $H \neq 1/2$ and in the mean squared if $H = 1/2$. Therefore except of the case $H = 1/2$ (ordinary Brownian motion or Wiener process) fractional Brownian motion cannot form a semimartingale so that classical stochastic integration does not work. However, the

Hölder continuity of B^H ensures the pathwise existence of our integrals (22)

$$\int_0^t f(s)\, dB^H(s),\ 0 < t \le T\ , \tag{33}$$

with probability 1 for any measurable random function f on $[0, T]$ such that $f_{0+} \in I^{\alpha}_{0+}(L_1(0, T))$ with probability 1 for some $\alpha > 1 - H$. Note, that we do not need here any assumption of adaptedness. In the special case $f \in H^{\lambda}(0, T)$ with probability 1 for some $\lambda > 1 - H$ we may use the interpretation as Riemann–Stieltjes integral and exploit the change-of-variable formula (31), the Hölder continuity of the integral as function of the boundary 4.4.1 and the integration rule 4.4.2. In particular, we may choose $f(t) = \sigma(X(t), t)$ for some real-valued Lipschitz function σ and any random function X whose sample paths lie in $H^{\lambda}(0, T)$ with probability 1. For $H > 1/2$ this makes it possible to investigate (stochastic) differential equations.

Example. Consider the linear equation

$$dX(t) = a\, X(t)\, dB^H(t) + b\, X(t)\, dt \tag{34}$$

which means

$$X(t) = X(0) + a \int_0^t X(s)\, dB^H(s) + b \int_0^t X(s)\, ds$$

for some random constants a and b, where $H > 1/2$.
Its unique solution reads

$$X(t) = X(0) \exp\{aB^H(t) + bt\} \tag{35}$$

Proof. The change-of-variable formula (31) implies that (35) is a solution of (34). Let $Y(t)$ be any other solution as above with the same initial condition $Y(0) = X(0)$. For simplicity we assume here that $X(0) \neq 0$ and show that Y agrees with X. (For the case $X(0) = 0$ this follows from a more general uniqueness result contained in a related Ph. D. Thesis which is in preparation.) We consider only (fixed) sample paths denoted by the same symbol Y which are Hölder continuous of order greater than $1/2$. In this case there are some numbers $C > c > 0$ such that $c < |(t)| < C$ and $\mathrm{sgn} Y(t) = \mathrm{sgn} Y(0)$ for $0 < t \le \epsilon$ with sufficiently small $\epsilon > 0$. For these t we may apply Theorem 4.3.1 to a smooth function F with $F(x) = \ln\, x$ if $x \in (c, C)$ and to $f(t) = |Y(t)|$ if $t \in [0, \epsilon]$ and obtain $\ln |Y(t)| - \ln |Y(0)| = aB(t) + bt$ according to Theorem 4.4.2. This yields $Y(t) = X(t)$ for $0 \le t \le \epsilon$. In the same way one can show that for any $t > 0$ with $Y(t) = X(t)$ there exists a right-sided neighborhood where the functions coincide.

We next consider the following example for the application of the change-of-variable formula 4.3.1:

$$\int_x^y B^H(t)\, dB^H(t) = \frac{1}{2}\, (B^H(y)^2 - B^H(x)^2), \quad 0 \le x < y < \infty \ , \tag{36}$$

with probability 1 provided that $H > 1/2$. This reflects the fact that the quadratic veriation of B^H vanishes. Note that for $H = 1/2$ in the exponent of (35) as well as on the right-hand side of (36) an additional linear term arises. Here the stochastic integrals are determined in the Itô sense. A link between both these approaches will be established in the next section.

5.2 A new representation of the Itô integral for random functions from $I_{0+}^{\alpha}(L_2)$

In this section the integrator g is replaced by the Wiener process $W = B^{1/2}$ and the random integrand f is assumed to be adapted with respect to the filtration given by W. If $f \in L_2(0,T)$ with probability 1 the classical Itô integral

$$\mathbf{I}_t f = (\text{Itô }) \int_0^t f\, dW$$

is determined. We write $\mathbf{I}f = \mathbf{I}_T f$.

5.2.1. Theorem. *If f is adapted and $f \in I_{0+}^{\alpha}(L_2)$ with probability* 1 *for some $\alpha > 1/2$ then the integrals $\int_0^t f dW$, $0 < t < T$, in the sense of* (22) *are determined and agree with the continuous version of the Itô integrals with probability* 1.

Proof. For the special case of smooth f both the integrals agree with

$$-\int_0^t f'(s) W(s)\, ds + f(t) W(t) \ .$$

Arbitrary realizations $f \in I_{0+}^{\alpha}(L_2)$ will again be approximated by the smooth functions $f_N = f * k_N$ with the smoothing kernels k_N given by (27). Using that $W \in I_{t-}^{1-\alpha}(L_2)$ with probability 1 we obtain from the proof of Theorem 2.4

$$\lim_{N\to\infty} \int_0^t f_N\, dW = \int_0^t f\, dW, \quad 0 < t < T \ ,$$

with probability 1. On the other hand the almost sure $L_2(0,T)$-convergence of f_N to f implies the convergence of $\mathbf{I}_t f_N$ to $\mathbf{I}_t f$ in probability for any fixed t. This yields the assertion. □

For applications to stochastic differential equation in the Itô sense the choice $\alpha > 1/2$ in Theorem 5.2.1 is too restrictive. Since the sample paths of W do not belong to $I_{0+}^{1/2}(L_2)$ the approach (22) does not work for $\alpha = 1/2$. However, we may approximate the Itô integrals of a function f with "fractional degree of differentiability" 1/2 by our integrals (22) for some regularization of f:

5.2.2. Corollary. *Let f be an adapted random function such that $f \in I_{0+}^{\alpha}(L_2)$ for any $\alpha < 1/2$ with probability* 1. *Then we have the following convergence in probability*:

$$\lim_{\epsilon \searrow 0} \int_0^t I_{0+}^{\epsilon} f \, dW = \lim_{\epsilon \searrow 0} (-1)^{1/2} \int_0^t D_{0+}^{1/2-\epsilon/2} f(s) D_{t-}^{1/2-\epsilon/2} W_{t-}(s) \, ds$$
$$= (\text{Itô }) \int_0^t f \, dW \ .$$

Proof. First note that $(I_{0+}^{\epsilon} f)_{0+} = I_{0+}^{\epsilon} f \in I_{0+}^{\alpha}(L_2)$ for any $1/2 < \alpha < 1/2 + \epsilon$. Therefore $\int_0^t I_{0+}^{\epsilon} f \, dW$ is determined by

$$(-1)^{1/2-\epsilon/2} \int_0^t D_{0+}^{1/2+\epsilon/2} I_{0+}^{\epsilon} f(s) D_{t-}^{1/2-\epsilon/2} W_{t-}(s) \, ds$$
$$= (-1)^{1/2-\epsilon/2} \int_0^t D_{0+}^{1/2-\epsilon/2} f(s) D_{t-}^{1/2-\epsilon/2} W_{t-}(s) \, ds \ .$$

According to Theorem 5.2.1 we have

$$\int_0^t I_{0+}^{\epsilon} f \, dW = \mathbf{I}_t (I_{0+}^{\epsilon} f)$$

with probability 1 for any $\epsilon > 0$. The almost sure $L_2(0,t)$-convergence of $I_{0+}^{\epsilon} f$ to f as $\epsilon \searrow 0$ implies the convergence in probability of the Itô integrals $\mathbf{I}_t(I_{0+}^{\epsilon} f)$ to $\mathbf{I}_t f$. □

Next we will state a sufficient condition for the above convergence in terms of square means. For, we introduce the classes $\mathbf{I}_{0+}^{\alpha}(\mathbf{L}_2)$ of measurable random functions f such that

$$\mathbf{E} f(0+)^2 < \infty \tag{37}$$

$$\mathbf{E} \int_0^T \frac{(f(t) - f(0+))^2}{t^{2\alpha}} \, dt < \infty \tag{38}$$

and the random functions

$$h_\epsilon(t,\omega) := \int_0^{t-\epsilon} \frac{f_{0+}(t,\omega) - f_{0+}(s,\omega)}{(t-s)^{\alpha+1}}\, ds \tag{39}$$

converge in $\mathbf{L}_2 := L_2((0,T)\times\boldsymbol{\Omega},\ \mathfrak{L}\times\mathbf{P})$ as $\epsilon \searrow 0$.

Note that (37) and (38) imply

$$\mathbf{E}\int_0^T f(t)^2\, dt < \infty. \tag{40}$$

We put

$$\mathbf{I}_{0+}^{\beta-}(\mathbf{L}_2) := \bigcap_{\alpha<\beta} \mathbf{I}_{0+}^{\alpha}(\mathbf{L}_2). \tag{41}$$

5.2.3. Corollary. *For any adapted* $f \in \mathbf{I}_{0+}^{1/2-}(\mathbf{L}_2)$ *we have*

$$\mathbf{E}\left(\int_0^t I_{0+}^{\epsilon} f\, dW - \mathbf{I}_t f\right)^2 = \mathbf{E}\int_0^t \left(I_{0+}^{\epsilon} f(s) - f(s)\right)^2 ds$$

which converges to zero as $\epsilon \searrow 0$.

Proof. Completely analogous arguments as in the deterministic case (s. [11], Theorem 13.2) show that $f \in \mathbf{I}_{0+}^{\alpha}(\mathbf{L}_2)$ implies $f_{0+} \in I_{0+}^{\alpha}(L_2)$ (and hence $f \in I_{0+}^{\alpha}(L_2)$ if $\alpha < 1/2$) with probability 1. Since the spaces $I_{0+}^{\alpha}(L_2)$ are decreasing in α it follows that for any $f \in \mathbf{I}_{0+}^{1/2-}(\mathbf{L}_2)$ we have $f \in \bigcap_{\alpha<1/2} I_{0+}^{\alpha}(L_2)$ with probability 1. Consequently, the conditions of Corollary 5.2.2 are fulfilled and

$$\int_0^t I_{0+}^{\epsilon} f\, dW = \mathbf{I}_t(I_{0+}^{\epsilon} f)$$

with probability 1. The $\mathbf{L}_2$-property (40) of f and the isometric behaviour of the Itô integral yield the asserted equation. The convergence

$$\lim_{\epsilon\searrow 0} \mathbf{E}\int_0^t (I_{0+}^{\epsilon} f(s) - f(s))^2\, ds = 0$$

may be shown similarly as in the deterministic case. □

Note that the Wiener process itself is an element of $\mathbf{I}_{0+}^{1/2-}(\mathbf{L}_2)$. Moreover, we have the following.

5.2.4. Theorem. *If f is adapted and* $f \in \mathbf{I}_{0+}^{1/2-}(\mathbf{L}_2)$ *then any measurable version* $X(t)$ *of the Itô-integral* (Itô) $\int_0^t f\, dW$, $t \in (0,T)$, *is again an element of* $\mathbf{I}_{0+}^{1/2-}(\mathbf{L}_2)$.

Proof. Let $0 < \alpha < 1/2$. The isometry property of the Itô integral leads to

$$\mathbf{E}\int_0^T X(t)t^{-2\alpha}\,dt = \int_0^T t^{-2\alpha}\mathbf{E}\int_0^t f(s)^2\,ds\,dt$$
$$= \mathbf{E}\int_0^T f(s)^2\int_s^T t^{-2\alpha}\,dt\,ds$$
$$\leq \text{const }\mathbf{E}\int_0^T f(t)^2\,dt < \infty\ .$$

It remains to prove that

$$\mathbf{E}\int_0^T\left(\int_{t-\epsilon}^{t-\epsilon'}\frac{X(t)-X(s)}{(t-s)^{\alpha+1}}\,ds\right)^2 dt$$

tends to zero as $\epsilon \searrow 0$ uniformly in $\epsilon' < \epsilon$. Here we put $X(s) = 0$ if $s < 0$. This expression equals

$$\int_0^T \mathbf{E}\left(\int_{\epsilon'}^{\epsilon}\frac{X(t)-X(t-s)}{s^{\alpha+1}}\,ds\right)^2 dt$$
$$= 2\int_0^T\int_{\epsilon'}^{\epsilon}\int_{\epsilon'}^{u} s^{-(\alpha+1)}u^{-(\alpha+1)}\mathbf{E}(X(t)$$
$$- X(t-s))(X(t)-X(t-u))\,ds\,du\,dt$$
$$= 2\int_0^T\int_{\epsilon'}^{\epsilon}\int_{\epsilon'}^{u} s^{-(\alpha+1)}u^{-(\alpha+1)}\mathbf{E}\int_{t-s}^{t} f(v)^2\,dv\,ds\,du\,dt$$

in view of the almost sure equality

$$X(t) - X(t-s) = (\text{Itô })\int_0^T 1_{(t-s,t)}(x)f(x)\,dW(x)$$

and the isometry property of the Itô integral. The right-hand side is equal to

$$2\int_{\epsilon'}^{\epsilon}\int_{\epsilon'}^{u} s^{-(\alpha+1)}u^{-(\alpha+1)}s\,ds\,du\,\mathbf{E}\int_0^T f(v)^2\,dv$$
$$\leq \frac{2}{1-\alpha}\int_{\epsilon'}^{\epsilon} u^{-(\alpha+1)}u^{1-\alpha}\,du\,\mathbf{E}\int_0^T f(v)^2\,dv \leq \text{const }\epsilon^{1-2\alpha}$$

which yields the assertion. □

5.3. Anticipating integrals

Our aim is now to extend the results of the preceding section to anticipating (i.e. non-adapted) functions f, where the Itô integral is extended to a new version of stochastic integral. This concept is closely related to Skorohod and extended Stratonovitch integrals. (For introduction, survey and further literature to related stochastic calculus cf. Nualart [6], Nualart and Pardoux [8], Pardoux [9].) Our main tool will be the classical approach to Skorohod integration (see Skorohod [12]) via Itô-Wiener chaos expansion of random $\mathbf{L}_2$-functions (see Itô [3]).

We first will establish a link between our integrals in the sense of (22) and the symmetric multiple Itô-integrals of deterministic functions $f \in L_2((0,1)^n)$ arising in the Itô-Wiener chaos expansion. The *iterated Itô integral* of such an f is given by

$$\mathbf{I}^n f := \text{ (Itô)} \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \ldots, t_n)\, dW(t_1) \ldots dW(t_n) \ .$$

(It is well-determined for tensor products $f = f_1 \otimes \cdots \otimes f_n$ and may be extended to general f by linearity and the corresponding isometry.) By means of symmetrization

$$\widetilde{f}(t_1, \ldots, t_n) := \frac{1}{n!} \sum_{\pi \in S_n} f(t_{\pi(1)}, \ldots, t_{\pi(n)})$$

(for the permutation group S_n) one turns to the concept of *symmetric n-th order Itô integral*

$$\widetilde{\mathbf{I}}^n f := n!\, \mathbf{I}^n \widetilde{f} \ . \tag{42}$$

Its isometry property reads

$$\mathbf{E}\widetilde{\mathbf{I}}^m f \widetilde{\mathbf{I}}^n g = \begin{cases} n! \int_{(0,1)^n} \widetilde{f}(t)\widetilde{g}(t) \mathfrak{L}^n(dt) & \text{if } m = n \\ 0, \text{ else } . \end{cases} \tag{43}$$

For $0 < \alpha < 1$ let $\widetilde{I}^{\alpha}_{0+}(n, L_2)$ be the class of those functions f from $L_2((0,1)^{n+1})$ being symmetric in the first n arguments for which the functions

$$h_\epsilon(t_1, \ldots, t_n, t) := \int_0^{t-\epsilon} \frac{f(t_1, \ldots, t_n, t) - f(t_1, \ldots, t_n, s)}{(t-s)^{\alpha+1}}\, ds \tag{44}$$

converge in $L_2((0,1)^{n+1})$ as $\epsilon \searrow 0$, where $h_\epsilon(t_1, \ldots, t_n, t) = 0$ if $t < 0$.

(Completely analogous arguments as in the proof of Theorem 13.2 in [11] show that $\widetilde{I}^{\alpha}_{0+}(n, L_2)$ is exactly the class of those functions on

$(0,1)^{n+1}$ with the required symmetry which are representable as I^{α}_{0+}-integral with respect to the last variable of some $L_2((0,1)^{n+1})$-function. Moreover, if $\alpha > 1/2$ these functions (up to equivalence) are Hölder continuous in the last argument, cf. [11], Theorem 3.6. Put

$$f_{0+}(t_1,\dots,t_n,t) := 1_{(0,1)}(t)(f(t_1,\dots,t_n,t) - (f(t_1,\dots,t_n,0+))$$

assuming everywhere that the right-sided limit exists.

For $f_{0+} \in \widetilde{I}^{\alpha}_{0+}(n,L_2)$ the symmetric multiple Itô integrals $\widetilde{\mathbf{I}}^{n-1}f(\cdot,s,t)$, $\widetilde{\mathbf{I}}^{n}f(\cdot,t)$, and $\widetilde{\mathbf{I}}^{n+1}f$ make sense because of the L_2-properties. In view of the isometry (43) the random functions $\widetilde{\mathbf{I}}^{n}f(\cdot,t)_{0+} = \widetilde{\mathbf{I}}^{n}f_{0+}(\cdot,t)$ are elements of the class $\mathbf{I}^{\alpha}_{0+}(\mathbf{L}_2)$ introduced in Section 5.2. Therefore the integrals $\int_0^1 \widetilde{\mathbf{I}}^{n}f(\cdot,t)\,dW(t)$ in the sence of (22) are determined with probability 1 for any $\alpha > 1/2$.

5.3.1. Theorem.

$$\int_0^1 \widetilde{\mathbf{I}}^{n}f(\cdot,t)\,dW(t) = \widetilde{\mathbf{I}}^{n+1}f + n\int_0^1 \widetilde{\mathbf{I}}^{n-1}f(\cdot,t,t)\,dt$$

with probability 1 *if* $f_{0+} \in \widetilde{I}^{\alpha}_{0+}(n,L_2)$ *for some* $\alpha > 1/2$.

Proof. By definition we have with probability 1

$$\int_0^1 \widetilde{\mathbf{I}}^{n}f(\cdot,t)\,dW(t) = (-1)^{\alpha}\int_0^1 D^{\alpha}_{0+}\widetilde{\mathbf{I}}^{n}f_{0+}(\cdot,\cdot)(t)D^{1-\alpha}_{1-}W_{1-}(t)\;dt \\ + \widetilde{\mathbf{I}}^{n}f(\cdot,0+)W(1)\;\;.$$

As before, we approximate f by functions being smooth in the last argument

$$f_N(t_1,\dots,t_n,t) := f(t_1,\dots,t_n,\cdot) * k_N(t)$$

and obtain by (43)

$$\widetilde{\mathbf{I}}^{n}f_N(\cdot,t) = \widetilde{\mathbf{I}}^{n}f(\cdot,\cdot) * k_N(t)\;\;,$$
$$\widetilde{\mathbf{I}}^{n-1}f_N(\cdot,s,t) = \widetilde{\mathbf{I}}^{n-1}f(\cdot,s,\cdot) * k_N(t)\;\;.$$

Then we have

$$\operatorname*{l.i.m.}_{N\to\infty}\int_0^1 \widetilde{\mathbf{I}}^{n}f_N(\cdot,t)\,dW(t) = \int_0^1 \widetilde{\mathbf{I}}^{n}f(\cdot,t)\,dW(t)$$

(cf. Section 2) and by (43)

$$\operatorname*{l.i.m.}_{N\to\infty}\widetilde{\mathbf{I}}^{n+1}f_N = \widetilde{\mathbf{I}}^{n+1}f$$

and similar estimates as in the proof of Theorem 3.6 in [11] yield

$$\underset{N\to\infty}{\text{l.i.m.}} \int_0^1 \widetilde{\mathbf{I}}^{n-1} f_N(\cdot, t, t)\, dt = \int_0^1 \widetilde{\mathbf{I}}^{n-1} f(\cdot, t, t)\, dt$$

(where l.i.m. means convergence in the mean square) since the smoothing kernels k_N are concentrated on $\left[0, \frac{1}{N}\right]$ and f is continuous in the last argument. Therefore is enough to consider f_N instead of f. We split the multiple integral on the left-hand side into a part not containing "diagonal" arguments and the remainder and approximate both the summands by piecewise integration:

$$\begin{aligned}
&\int_0^1 \widetilde{\mathbf{I}}^n f_N(\cdot, t)\, dW(t) \\
&= n!\, \underset{k\to\infty}{\text{l.i.m.}} \sum_{j=0}^{n} \sum_{0\le i_1<\cdots<i_j<i<i_{j+1}<\cdots<i_n\le k} \\
&\qquad \int_{\frac{i}{k}}^{\frac{i+1}{k}} (\text{Itô}) \int_{\frac{i_n}{k}}^{\frac{i_n+1}{k}} \cdots \int_{\frac{i_1}{k}}^{\frac{i_1+1}{k}} f_N(t_1, \ldots, t_n, t)\, dW(t_1) \ldots\, dW(t_n)\, dW(t) \\
&\quad + n!\, \underset{k\to\infty}{\text{l.i.m.}} \sum_{j=1}^{n} \sum_{0\le i_1<\cdots<i_n\le k} \\
&\qquad \int_{\frac{i_j}{k}}^{\frac{i_j+1}{k}} (\text{Itô}) \int_{\frac{i_n}{k}}^{\frac{i_n+1}{k}} \cdots \int_{\frac{i_1}{k}}^{\frac{i_1+1}{k}} f_N(t_1, \ldots, t_n, t)\, dW(t_1) \ldots\, dW(t_n)\, dW(t)
\end{aligned}$$

Because of the choice of disjoint intervals of integration and the independent increments of the Wiener process we may change the order of integration in the first summand and exploit that

$$\int_{\frac{i}{k}}^{\frac{i+1}{k}} f_N(t_1, \ldots, t_n, t)\, dW(t)$$

agrees with the Itô integral in the sense of the corresponding $\mathbf{L}_2$-equality. Therefore the first limit yields $\widetilde{\mathbf{I}}^{n+1} f_N$ because of the symmetry of f_N in the first n arguments. In the second summand we also may change the order of integration except of the "diagonal" integrals arising from

$$\int_{\frac{i_j}{k}}^{\frac{i_j+1}{k}} (\text{Itô }) \int_{\frac{i_j}{k}}^{\frac{i_j+1}{k}} f_N(t_1, \ldots, t_n, t)\, dW(t_j)\, dW(t) \ .$$

By the smoothness property of f_N we may use formula (22) with $\alpha = 1$ for the outer integral to obtain the expression

$$\begin{aligned}
&- \int_{\frac{i_j}{k}}^{\frac{i_j+1}{k}} (\text{Itô }) \int_{\frac{i_j}{k}}^{\frac{i_j+1}{k}} f_N'(t_1, \ldots, t_n, t)\, dW(t_j) W_{\frac{i_j+1}{k}-}(t)\, dt \\
&\quad + (\text{Itô }) \int_{\frac{i_j}{k}}^{\frac{i_j+1}{k}} f_N(t_1, \ldots, t_n, \frac{i_j}{k})\, dW(t_j) \left(W\left(\frac{i_j+1}{k}\right) - W\left(\frac{i_j}{k}\right)\right) \ .
\end{aligned}$$

The first summand may be neglected asymptotically as $k \to \infty$ after integrating in $t_l, l \neq j$, and summing up in view of the usual $\mathbf{L}_2$-estimations. Similar $\mathbf{L}_2$-arguments show that the second summand may be replaced asymptotically by

$$f_N\left(t_1, \ldots, t_{j-1}, \frac{i_j}{k}, t_{j+1}, \ldots, t_n, \frac{i_j}{k}\right)\left(W\left(\frac{i_j+1}{k}\right) - W\left(\frac{i_j}{k}\right)\right)^2 .$$

Using the quadratic variation of the Wiener process and the symmetry property of f_N we obtain after integration and summation for the corresponding limit the value

$$n\int_0^1 \widetilde{\mathbf{I}}^{n-1} f_N(\cdot, t, t)\, dt \ .$$

□

We now turn to anticipating integrals using the *Itô–Wiener chaos expansion* of random functions $f \in \mathbf{L}_2$:

$$f(t) = \sum_{n=0}^{\infty} \widetilde{\mathbf{I}}^n f^n(\cdot, t) \tag{45}$$

(where $\widetilde{\mathbf{I}}^0 f^0(\cdot, t) = \mathbf{E}f(t)$) for unique $f^n \in L_2((0,1)^{n+1})$ being symmetric in the first n-arguments. Recall that the *Skorohod integral* of f exists and is given by

$$\delta(f) = (S)\int_0^1 f\, dW := \sum_{n=0}^{\infty} \widetilde{\mathbf{I}}^{n+1} f^n \tag{46}$$

if this series converges in the mean square.

We introduce the *extended Stratonovitch integral* of f by

$$\int_0^1 f \circ dW := \sum_{n=0}^{\infty}\left(\widetilde{\mathbf{I}}^{n+1} f^n + \frac{n}{2}\int_0^1 (\widetilde{\mathbf{I}}^{n-1} f^n(\cdot, t, t-) + \widetilde{\mathbf{I}}^{n-1} f^n(\cdot, t, t+))\, dt\right) \tag{47}$$

if this series converges in the mean square. One can show that under some additional condition this integral agrees with the notion used in the literature (cf. [6]).

Theorem 5.3.1 suggests the following new concept of *anticipating integral*:

$$(A)\int_0^1 f\, dW := \sum_{n=0}^{\infty}\left(\widetilde{\mathbf{I}}^{n+1} f^n + n\int_0^1 \widetilde{\mathbf{I}}^{n-1} f^n(\cdot, t, t-)\, dt\right) \tag{48}$$

provided that the series converges in the mean square, where

$$\int_0^1 \widetilde{I}^{n-1} f^n(\cdot, t, -t)\, dt = \underset{\epsilon \searrow 0}{\text{l.i.m.}} \int_0^1 \widetilde{I}^{n-1} I^{\epsilon}_{0+} f^n(\cdot, t, t)\, dt \ .$$

(The corresponding expression for $t+$ in (47) is defined similarly.)

If the f^n (up to $L_2((0,1)^{n-1})$-equivalence) have no jumps at Lebesgue-a.a. points on the diagonal given by the last two arguments we get

$$(A) \int_0^1 f\, dW = \int_0^1 f \circ\, dW \ .$$

For adapted f we have $f^n(\cdot, t, t-) = 0$ at a.a. t and therefore

$$(A) \int_0^1 f\, dW = \delta(f) = (\text{Itô}) \int_0^1 f\, dW$$

with probability 1. (The last equation is the well-known extension property of the Skorohod integral.)

In general, *the three integrals are different* and the existence of the Stratonovitch integral or the anticipating integral (48) does not imply that of the Skorohod integral. Our definition will be justified below where we will study some relationships between these integrals.

It appears appropriate to work with the following *Slobodeckij-type spaces* $\mathbf{W}^{\alpha}_{2,+} = \mathbf{W}^{\alpha}_{2,+}(0,1)$ of measurable random functions f such that

$$\mathbf{E}\, f(0+)^2 < \infty \tag{49}$$

$$\mathbf{E} \int_0^1 \frac{(f(t) - f(0+))^2}{t^{2\alpha}}\, dt < \infty \tag{50}$$

$$\mathbf{E} \int_0^1 \int_0^1 \frac{(f(t) - f(s))^2}{|t-s|^{2\alpha+1}}\, ds\, dt < \infty \ , \tag{51}$$

where $0 < \alpha < 1$. (For a survey on classical function spaces see, e.g., [7].)

Remark: (49) and (50) imply

$$\mathbf{E} \int_0^1 f(t)^2\, dt < \infty \ . \tag{52}$$

In the theory of function spaces the deterministic versions of (51) and (52) define the space $W_2^{\alpha}(0,1)$.

Recall that the space $\mathbf{I}^{\alpha-}_{0+}(\mathbf{L}_2)$ was defined in (41).

5.3.2. Proposition.

$$\mathbf{W}^{\alpha}_{2,+} \subset \mathbf{I}^{\alpha-}_{0+}(\mathbf{L}_2) \ .$$

Proof. According to (39) ist is enough to show that for any $\beta < \alpha$ we have

$$\lim_{\epsilon \searrow 0} \sup_{\epsilon' < \epsilon} \mathbf{E} \int_0^1 \left(\int_{t-\epsilon}^{t-\epsilon'} \frac{f_{0+}(t) - f_{0+}(s)}{(t-s)^{\beta+1}} \right)^2 dt = 0 \ .$$

Choose $0 < \delta < 2(\alpha - \beta)$. By the Chauchy–Schwarz inequality we obtain

$$\begin{aligned}
&\mathbf{E} \int_0^1 \left(\int_{t-\epsilon}^{t-\epsilon'} \frac{f_{0+}(t) - f_{0+}(s)}{(t-s)^{\beta+1}} \right)^2 dt \\
&\quad = \mathbf{E} \int_0^1 \left(\int_{t-\epsilon}^{t-\epsilon'} \frac{f_{0+}(t) - f_{0+}(s)}{(t-s)^{\beta+\delta}} \frac{1}{(t-s)^{1-\delta}} \right)^2 dt \\
&\quad \le \text{const } \mathbf{E} \int_0^1 \int_{t-\epsilon}^{t-\epsilon'} \frac{(f_{0+}(t) - f_{0+}(s))^2}{(t-s)^{2\beta+2\delta}} \frac{1}{(t-s)^{1-\delta}} \, ds \, dt \\
&\quad = \text{const } \mathbf{E} \int_0^1 \int_{t-\epsilon}^{t-\epsilon'} \frac{(f_{0+}(t) - f_{0+}(s))^2}{(t-s)^{2\alpha+1}} (t-s)^{2(\alpha-\beta)-\delta} \, ds \, dt \\
&\quad \le \text{const } (\epsilon - \epsilon')^{2(\alpha-\beta)-\delta} \left(\mathbf{E} \int_0^1 \int_0^1 \frac{(f(t) - f(s))^2}{|t-s|^{2\alpha+1}} \, ds \, dt \right. \\
&\qquad \left. + \mathbf{E} \int_0^\epsilon f_{0+}(t)^2 \int_{t-\epsilon}^0 \frac{1}{(t-s)^{2\alpha+1}} \, ds \, dt \right) \\
&\quad \le \text{const} \epsilon^{2(\alpha-\beta)-\delta} \left(\text{const } + \mathbf{E} \int_0^\epsilon \frac{(f(t) - f(0+))^2}{t^{2\alpha}} \, dt \right) \\
&\quad \le \text{const } \epsilon^{2(\alpha-\beta)-\delta} \ ,
\end{aligned}$$

where the constants depend on δ. □

Let now $f(t) = \sum_{n=0}^{\infty} \widetilde{\mathbf{I}}^n f^n(\cdot, t)$ be the Itô–Wiener chaos expansion as above.

5.3.3. Proposition $f \in \mathbf{W}_{2,+}^{\alpha}$ *implies* $f_{0+}^n \in \widetilde{I}_{0+}^{\beta}(n, L_2)$ *for any* $\beta < \alpha$ *and* $n \in \mathbb{N}$.

Proof. By the definition of $\widetilde{I}_{0+}^{\beta}(n, L_2)$ (cf. (44)) it suffices to show that

$$\lim_{\epsilon \searrow 0} \sup_{\epsilon' < \epsilon} \int_0^1 \cdots \int_0^1 \left(\int_{t-\epsilon}^{t-\epsilon'} \frac{f_{0+}^n(t_1, \ldots, t_n, t) - f_{0+}^n(t_1, \ldots, t_n, s)}{(t-s)^{\beta+1}} \, ds \right)^2 dt \, dt_1 \ldots dt_n = 0 \ .$$

Similar estimations like in the proof of the preceding proposition lead to the following upper bound of these integrals for fixed $0 < \delta < 2(\alpha - \beta)$:

$$\text{const } \epsilon^{2(\alpha-\beta)-\delta}\left(\int_0^1 \cdots \int_0^1 \frac{(f^n(t_1,\ldots,t_n,t)-f_n(t_1,\ldots,t_n,0+))^2}{t^{2\alpha}} dt_1\ldots dt_n\, dt + \int_0^1 \cdots \int_0^1 \frac{(f^n(t_1,\ldots,t_n,t)-f_n(t_1,\ldots,t_n,s))^2}{|t-s|^{2\alpha+1}} dt_1\ldots dt_n\, ds\, dt\right) .$$

It remains to show that the last two integrals are finite. Using the isometry (43) we infer

$$\begin{aligned} n!\int_0^1 \cdots \int_0^1 &\frac{(f^n(t_1,\ldots,t_n,t)-f^n(t_1,\ldots,t_n,0+))^2}{t^{2\alpha}} dt_1\ldots dt_n\, dt \\ &= \int_0^1 t^{-2\alpha}\mathbf{E}(\widetilde{\mathbf{I}}^n(f^n(\cdot,t)-f^n(\cdot,0+)))^2\, dt \\ &\le \int_0^1 \frac{\mathbf{E}(f(t)-f(0+))^2}{t^{2\alpha}}\, dt\ , \end{aligned}$$

since $f(t)-f(0+)=\sum_{n=0}^{\infty}\widetilde{\mathbf{I}}^n(f^n(\cdot,t)-f^n(\cdot,0+))$ and hence, $\mathbf{E}(f(t)-f(0+))^2=\sum_{n=0}^{\infty}\mathbf{E}(\mathbf{I}^n(f^n(\cdot,t)-f^n(\cdot,0+)))^2$ by orthogonality. In view of (50) the last integral is finite. Similarly one shows that

$$\begin{aligned} n!\int_0^1 \cdots \int_0^1 &\frac{(f^n(t_1,\ldots,t_n,t)-f^n(t_1,\ldots,t_n,s))^2}{|t-s|^{2\alpha+1}} dt_1\ldots dt_n\, ds\, dt \\ &\le \int_0^1\int_0^1 \mathbf{E}\frac{(f(t)-f(s))^2}{|t-s|^{2\alpha+1}}\, ds\, dt \end{aligned}$$

which is finite according to (51). □

We now are able to prove an extension of Theorem 5.2.1 to anticipating functions which justifies the definition (48).

5.3.4. Theorem. *If* $f\in\mathbf{W}^{\alpha}_{2,+}$ *for some* $\alpha>1/2$ *then the anticipating integral* $(A)\int_0^1 f\, dW$ *in the sense of* (48) *exists and agrees with the integral* $\int_0^1 f\, dW$ *in the sense of* (22) *as well as with the extended Stratonovitch integral* $\int_0^1 f\circ dW$.

Proof. Choose an arbitrary $\beta\in(1/2,\alpha)$. By Proposition 5.3.2,

$$\int_0^1 f\, dW = (-1)^{\beta}\int_0^1 D^{\beta}_{0+}f_{0+}(t)D^{1-\beta}_{1-}W_1(t)\, dt + f(0+)W(1)\ .$$

Recall that

$$f_{0+}(t) = \sum_{n=0}^{\infty} \widetilde{\mathbf{I}}^n f_{0+}^n(\cdot, t), \quad f(0+) = \sum_{n=0}^{\infty} \widetilde{\mathbf{I}}^n f^n(\cdot, 0+)$$

and therefore,

$$(A) \int_0^1 f \, dW = (A) \int_0^1 f_{0+} \, dW + f(0+) W(1) \ .$$

If we can show that

$$D_{0+}^{\beta} f_{0+}(t) = \sum_{n=0}^{\infty} D_{0+}^{\beta} \widetilde{\mathbf{I}}^n f_{0+}^n(\cdot, \cdot)(t)$$

in the sense of $\mathbf{L}_2$-convergence of the series then the Cauchy–Schwarz inequality leads to

$$\int_0^1 f_{0+} \, dW = \sum_{n=0}^{\infty} \int_0^1 \widetilde{\mathbf{I}}^n f_{0+}^n(\cdot, t) \, dW(t)$$

so that Proposition 5.3.3 and Theorem 5.3.1 yield the assertion.

By construction, for any $h \in \mathbf{I}_{0+}^{\beta}(\mathbf{L}_2)$ the derivative $D_{0+}^{\beta} h_{0+}$ is the $\mathbf{L}_2$-limit of the random functions

$$D_{0+,\epsilon}^{\beta} h_{0+}(t) := \frac{1}{\Gamma(1-\beta)} \left(\frac{h(t) - h(0+)}{t^{\beta}} + \beta \int_0^{t-\epsilon} \frac{h_{0+}(t) - h_{0+}(s)}{(t-s)^{\beta+1}} \, ds \right)$$

as $\epsilon \searrow 0$. For the special functions

$$h_N(t) := \sum_{n=0}^{N} \widetilde{\mathbf{I}}^n f_{0+}^n(\cdot, t)$$

we obtain by Fubini and the orthogonality $\mathbf{E}\widetilde{\mathbf{I}}^n \varphi \widetilde{\mathbf{I}}^m \psi = 0$; $n \neq m$, the estimation

$$\begin{aligned}
&\int_0^1 \mathbf{E}(D_{0+,\epsilon}^{\beta} h_N(t) - D_{0+,\epsilon'}^{\beta} h_N(t))^2 \, dt \\
&\quad = \int_0^1 \sum_{n=0}^{N} \mathbf{E}(D_{0+,\epsilon}^{\beta} \widetilde{\mathbf{I}}^n f_{0+}^n(\cdot, \cdot)(t) - D_{0+,\epsilon'}^{\beta} \widetilde{\mathbf{I}}^n f_{0+}^n(\cdot, \cdot)(t))^2 \, dt \\
&\quad \le \int_0^1 \sum_{n=0}^{\infty} \mathbf{E}(D_{0+,\epsilon}^{\beta} \widetilde{\mathbf{I}}^n f_{0+}^n(\cdot, \cdot)(t) - D_{0+,\epsilon'}^{\beta} \widetilde{\mathbf{I}}^n f_{0+}^n(\cdot, \cdot)(t))^2 \, dt \ .
\end{aligned}$$

Moreover, for any $\epsilon > 0$,

$$D_{0+,\epsilon}^{\beta} f_{0+} = \sum_{n=0}^{\infty} D_{0+,\epsilon}^{\beta} \widetilde{\mathbf{I}}^n f_{0+}^n(\cdot, \cdot)$$

because of the corresponding boundedness property. Therefore the expression on the right-hand side of the above estimation is equal to

$$\int_0^1 \mathbf{E}(D^{\beta}_{0+,\epsilon} f_{0+}(t) - D^{\beta}_{0+,\epsilon'} f_{0+}(t))^2 \, dt \ .$$

In view of Proposition 5.3.2 the function f_{0+} is an element of $\mathbf{I}^{\beta}_{0+}(\mathbf{L}_2)$. Hence, the last integral tends to zero as $\epsilon \searrow 0$ uniformly in $\epsilon' < \epsilon$ and consequently, the $\mathbf{L}_2$-convergence of $D^{\beta}_{0+,\epsilon} h_N(t)$ as $\epsilon \searrow 0$ is uniform in N. Thus we may change the order of the $\mathbf{L}_2$-limits and obtain

$$\begin{aligned} D^{\beta}_{0+} f_{0+} &= \lim_{\epsilon \searrow 0} \lim_{N\to\infty} D^{\beta}_{0+,\epsilon} h_N = \lim_{N\to\infty} \lim_{\epsilon \searrow 0} D^{\beta}_{0+,\epsilon} h_N \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} D^{\beta}_{0+} \widetilde{\mathbf{I}}^n f^n_{0+}(\cdot,\cdot) = \sum_{n=0}^{\infty} D^{\beta}_{0+} \widetilde{\mathbf{I}}^n f^n_{0+}(\cdot,\cdot) \ . \end{aligned}$$

Finally, the equality

$$(A) \int_0^1 f \, dW = \int_0^1 f \circ dW$$

follows from the definition of the extended Stratonovitch integral and continuity of the functions f^n in the last argument because of Proposition 5.3.3. □

Recall that the condition $\alpha > 1/2$ is too restrictive concerning the application to stochastic differential equations. In order to extend Corollary 5.2.3 to anticipating f we introduce the class

$$\mathbf{W}^{1/2-}_{2,+} := \bigcap_{0<\alpha<1/2} \mathbf{W}^{\alpha}_{2,+} \ . \tag{53}$$

By Proposition 5.3.2 , $\mathbf{W}^{1/2-}_{2,+} \subset \mathbf{I}^{1/2-}_{2,+}(\mathbf{L}_2)$.

Similarly as in the deterministic case one proves that $f \in \mathbf{W}^{1/2-}_{2,+}$ implies $I^{\epsilon}_{0+} f \in \mathbf{W}^{(1/2+\epsilon)-}_{2,+}$. Therefore we may consider again the "approximating" integrals

$$\int_0^1 I^{\epsilon}_{0+} f \, dW = (-1)^{1/2-\epsilon/2} \int_0^1 D^{1/2-\epsilon/2}_{0+} f(t) D^{1/2-\epsilon/2}_{1-} W_{1-}(t) \, dt$$

as $\epsilon \searrow 0$. Theorem 5.3.4 implies

$$\int_0^1 I^{\epsilon}_{0+} f \, dW = (A) \int_0^1 I^{\epsilon}_{0+} f \, dW \ .$$

Since f is anticipating convergence in $\mathbf{L}_2$ of these integrals as $\epsilon \searrow 0$ does not hold in general. However, we get the following main result of this section regarding that the mean square of $(A) \int_0^1 f \, dW$ in the sense of (48) may be computed by

$$\left(\int_0^1 f^1(t,t-)\,dt\right)^2+\sum_{n=0}^{\infty}(n+1)!\Big\|\widetilde{f}^n$$
$$+(n+2)\int_0^1 f^{n+2}(\cdot,t,t-)\,dt\Big\|^2_{L_2((0,1)^{n+1})} \tag{54}$$

in view of the isometry property (43).

5.3.5. Theorem. *Suppose that* $f\in\mathbf{W}^{1/2-}_{2,+}$ and $\int_0^1 I^\epsilon_{0+}f\,dW$ *converges in the mean square as* $\epsilon\searrow 0$. *Then the anticipating integral* $(A)\int_0^1 f\,dW$ *in the sense of* (48) *exists and we have*

$$\mathbf{E}\left(\int_0^1 I^\epsilon_{0+}f\,dW-(A)\int_0^1 f\,dW\right)^2$$
$$=\left(\int_0^1 (I^\epsilon_{0+}f^1(t,t-)-f^1(t,t-))\,dt\right)^2+\sum_{n=0}^{\infty}(n+1)!$$
$$\Big\|I^\epsilon_{0+}\widetilde{f}^n-\widetilde{f}^n+(n+2)\int_0^1 (I^\epsilon_{0+}f^{n+2}(\cdot,t,t-)-f^{n+2}(\cdot,t,t-))\Big\|^2_{L_2((0,1)^{n+1}}$$

(*where* $f^k(\cdot,t,t-)$ *is the* L_2*-limit of* $I^\epsilon_{0+}f^k(\cdot,t,t)$ *as function in* t *as* $\epsilon\searrow 0$) *and*

$$\operatorname*{l.i.m.}_{\epsilon\searrow 0}\int_0^1 I^\epsilon_{0+}f\,dW=(A)\int_0^1 f\,dW\ .$$

Proof. Let $0<\epsilon<1/2$. It follows from the Cauchy–Schwarz inequality that

$$I^\epsilon_{0+}f=\sum_{n=0}^{\infty}I^\epsilon_{0+}\widetilde{\mathbf{I}}^n f^n(\cdot,\cdot)\ .$$

Then the isometry (43) yields the Itô–Wiener chaos expansion

$$I^\epsilon_{0+}f=\sum_{n=0}^{\infty}\widetilde{\mathbf{I}}^n I^\epsilon_{0+}f^n(\cdot,\cdot)\ .$$

Since $\int_0^1 I^\epsilon_{0+}f\,dW=(A)\int_0^1 I^\epsilon_{0+}f\,dW$ it is enough to prove that

$$\operatorname*{l.i.m.}_{\epsilon\searrow 0}\sum_{n=0}^{\infty}\left(\widetilde{\mathbf{I}}^{n+1}I^\epsilon_{0+}f^n+n\int_0^1\widetilde{\mathbf{I}}^{n-1}I^\epsilon_{0+}f^n(\cdot,t,t)\,dt\right)$$
$$=\sum_{n=0}^{\infty}\left(\widetilde{\mathbf{I}}^{n+1}f^n+n\int_0^1\widetilde{\mathbf{I}}^{n-1}f^n(\cdot,t,t-)\,dt\right)\ .$$

(The asserted equation for the mean square distance follows from (54).) The series on the left-hand side is equivalent to the series

$$\int_0^1 I_{0+}^{\epsilon} f^1(t,t)\, dt + \sum_{n=0}^{\infty} \left(\widetilde{\mathbf{I}}^{n+1} I_{0+}^{\epsilon} f^n + (n+2) \int_0^1 \widetilde{\mathbf{I}}^{n+1} I_{0+}^{\epsilon} f^{n+2}(\cdot,t,t)\, dt \right)$$

whose summands are pairwise orthogonal according to (43). By assumption, the limit in the mean square as $\epsilon \searrow 0$ exists. The Hilbert space arguments which we have used repeatedly show that this limit agrees with

$$\lim_{\epsilon \searrow 0} \int_0^1 I_{0+}^{\epsilon} f^1(t,t)\, dt + \sum_{n=0}^{\infty} \underset{\epsilon \searrow 0}{\text{l.i.m.}} \left(\widetilde{\mathbf{I}}^{n+1} I_{0+}^{\epsilon} f^n + (n+2) \right.$$

$$\left. \int_0^1 \widetilde{\mathbf{I}}^{n+1} I_{0+}^{\epsilon} f^{n+2}(\cdot,t,t)\, dt \right)$$

$$= \int_0^1 f^1(t,t-)\, dt + \sum_{n=0}^{\infty} \left(\widetilde{\mathbf{I}}^{n+1} f^n + (n+2) \int_0^1 \widetilde{\mathbf{I}}^{n+1} f^{n+2}(\cdot,t,t-)\, dt \right)$$

in view of the isometry property (43) and the corresponding L_2-versions of (18). Finally, the right-hand side is equivalent to

$$\sum_{n=0}^{\infty} \left(\widetilde{\mathbf{I}}^{n+1} f^n + n \int_0^1 \widetilde{\mathbf{I}}^{n-1} f^n(\cdot,t,t-)\, dt \right) .$$

□

5.3.6. Corollary. *Under the conditions of Theorem* 5.3.5 *we have*

$$(A) \int_0^1 cf\, dW = c\, (A) \int_0^1 f\, dW$$

for any bounded random variable c.

Proof. The definition (22) implies

$$\int_0^1 I_{0+}^{\epsilon}(cf)\, dW = c \int_0^1 I_{0+}^{\epsilon} f\, dW \; .$$

Therefore Theorem 5.3.6 yields the assertion. □

In order to formulate a certain counterpart to Theorem 5.3.6 we need some notions from the literature. Recall that in terms of Itô–Wiener chaos expansion $f(s) = \sum_{n=0}^{\infty} \widetilde{\mathbf{I}}^n f^n(\cdot,s)$ for fixed s the *Malliavin derivative* of this random variable is given by

$$D_t f(s) = \sum_{n=1}^{\infty} n\, \widetilde{\mathbf{I}}^{n-1} f^n(\cdot, t, s)$$

provided that this series of random functions in t converges in $\mathbf{L}_2$ (cf. [6], [8], [9]). The space $\mathbb{L}^{1,2}$ of random functions is commonly used in the literature in order to characterize the Skorohod integral as dual operation to Malliavin derivation. For its definition we refer to [9]. It is a subspace of the domain of definition of the Skorohod integral. $\mathbb{L}_C^{1,2}$ denotes the space of those $f \in \mathbb{L}^{1,2}$ for which the set of functions

$$\{s \to D_t f(s); \quad s \in [0,1] \setminus \{t\}\}_{t \in (0,1)}$$

is equicontinuous with values in $L_2(\Omega, \mathbf{P})$ and

$$\operatorname*{ess\,sup}_{(s,t) \in (0,1)^2} \mathbb{E}(D_t f(s))^2 < \infty \ .$$

For $f \in \mathbb{L}_C^{1,2}$ denote

$$D_t f(t-) := \operatorname*{l.i.m.}_{\epsilon \searrow 0} D_t f(t - \epsilon)$$

$$D_t f(t+) := \operatorname*{l.i.m.}_{\epsilon \searrow 0} D_t f(t + \epsilon)$$

(cf. [9]).

5.3.7. Theorem.

(i) *For $f \in \mathbb{L}_C^{1,2}$ the anticipating integral* (48) *exists and equals*

$$(A) \int_0^1 f \, dW = \delta(f) + \int_0^1 D_t f(t-)\, dt \ .$$

(ii) *If $f \in \mathbb{L}_C^{1,2} \cap \mathbf{W}_{2,+}^{1/2-}$ then the convergence*

$$\operatorname*{l.i.m.}_{\epsilon \searrow 0} \int_0^1 I_{0+}^{\epsilon} f \, dW = (A) \int_0^1 f \, dW$$

holds.

Remark. Similarly as in the proof below it can be shown that $f \in \mathbb{L}_C^{1,2}$ implies the existence of the extended Stratonovitch integral and

$$\int_0^1 f \circ \, dW = \delta(f) + \frac{1}{2} \int_0^1 (D_t f(t-) + D_t f(t+))\, dt \ .$$

(This corresponds to Proposition 5.2 in [9].)

Proof of Theorem 5.3.7. (i) In order to show the mean square convergence of the series

$$\sum_{n=0}^{\infty}\left(\widetilde{\mathbf{I}}^{n+1}f^n + n\int_0^1 \widetilde{\mathbf{I}}^{n-1}f^n(\cdot,t,t-)\,dt\right)$$

and the asserted equation we use the convergence

$$\sum_{n=0}^{\infty}\widetilde{\mathbf{I}}^{n+1}f^n = \delta(f)$$

and prove that

$$\sum_{n=0}^{\infty} n\int_0^1 \widetilde{\mathbf{I}}^{n-1}f^n(\cdot,t,t-)\,dt = \int_0^1 D_t f(t-)\,dt \ .$$

Regarding

$$\begin{aligned} D_t f(t-) &= \underset{\epsilon\searrow 0}{\text{l.i.m.}} \sum_{n=1}^{\infty} n\,\widetilde{\mathbf{I}}^{n-1}f^n(\cdot,t,t-\epsilon) \\ &= \sum_{n=1}^{\infty} n\,\widetilde{\mathbf{I}}^{n-1}f^n(\cdot,t,t-) \end{aligned}$$

at almost all t by (43) we still have to justify the change of the order of summation in n and integration in t. But this also follows from (43).
(ii) It is not difficult to check that $f \in \mathbf{L}_C^{1,2}$ implies $I_{0+}^{\epsilon}f \in \mathbf{L}_C^{1,2}$. Hence,

$$(A)\int_0^1 I_{0+}^{\epsilon}f\,dW = \delta(I_{0+}^{\epsilon}f) + \int_0^1 D_t I_{0+}^{\epsilon}f(t-)\,dt \ .$$

Further, the above representation of $D_t f(t-)$ in terms of the Itô–Wiener chaos expansion and (43) yield

$$D_t I_{0+}^{\epsilon}f(t-) = I_{0+}^{\epsilon}D_t f(\cdot)(t-) \ .$$

From the corresponding L_2-version of (18) we infer

$$\underset{\epsilon\searrow 0}{\text{l.i.m.}}\int_0^1 I_{0+}^{\epsilon}D_t f(\cdot)(t-)\,dt = \int_0^1 D_t f(t-)\,dt \ .$$

Below we will show that

$$\underset{\epsilon\searrow 0}{\text{l.i.m.}}\,\delta(I_{0+}^{\epsilon}f) = \delta(f) \ .$$

Consequently,

$$\underset{\epsilon\searrow 0}{\text{l.i.m.}}\,(A)\int_0^1 I_{0+}^{\epsilon}f\,dW = (A)\int_0^1 f\,dW \ .$$

If we additionally assume that $f \in \mathbf{W}_{2,+}^{1/2-}$ then we may use

$$\int_0^1 I_{0+}^{\epsilon}f\,dW = (A)\int_0^1 I_{0+}^{\epsilon}f\,dW$$

in view of Theorem 5.3.4. This leads to (ii).

By definition of the Skorohod integral,

$$\delta(I_{0+}^{\epsilon} f) = \sum_{n=0}^{\infty} \widetilde{\mathbf{I}}^{n+1}(I_{0+}^{\epsilon} f^n)$$

$$= \sum_{n=0}^{N} \widetilde{\mathbf{I}}^{n+1}(I_{0+}^{\epsilon} f^n) + \sum_{n=N+1}^{\infty} \widetilde{\mathbf{I}}^{n+1}(I_{0+}^{\epsilon} f^n) \ .$$

For fixed N the first summand tends to $\sum_{n=0}^{N} \widetilde{\mathbf{I}}^{n+1}(f)$ as $\epsilon \searrow 0$ by (43) and the corresponding L_2-version of (18). The mean square of the second summand does not exceed

$$\sum_{n=N+1}^{\infty} (n+1)! \| I_{0+}^{\epsilon} \widetilde{f}^n \|^2 \le \text{const} \sum_{n=N+1}^{\infty} (n+1)! \| \widetilde{f}^n \|^2$$

for a certain constant independent of ϵ because

$$\| I_{0+}^{\epsilon} \widetilde{f}^n \|^2 = \int_0^1 \cdots \int_0^1 \left(\int_0^t \widetilde{f}^n(t_1, \ldots, t_n, s) \frac{1}{\Gamma(\epsilon)} (t-s)^{\epsilon-1} \, ds \right)^2 dt \, dt_1 \ldots dt_n$$

$$\le \text{const} \int_0^1 \cdots \int_0^1 \int_0^t \widetilde{f}^n(t_1, \ldots, t_n, s)^2 \frac{1}{\Gamma(\epsilon)} (t-s)^{\epsilon-1} \, ds \, dt \, dt_1 \ldots dt_n$$

$$\le \text{const} \ \| \widetilde{f}^n \|^2$$

according to the Cauchy–Schwarz inequality.

Since $\|\delta(f)\|^2 = \sum_{n=0}^{\infty} (n+1)! \ \|\widetilde{f}^n\|^2$ we obtain that the second summand of the above sum tends to zero as $N \to \infty$ uniformly in ϵ. Thus,

$$\underset{\epsilon \searrow 0}{\text{l.i.m.}} \ \delta(I_{0+}^{\epsilon} f) = \delta(f) \ . \qquad \square$$

Remark. 1. Recall that under various conditions an extended Itô formula for the change of variables in Skorohod integrals was proved. In distinction to the adapted case it contains an additional term concerning Malliavin derivatives. (For Stratonovitch integrals the classical chain rule from calculus remains valid.) We will show in part II of this paper that under appropriate conditions for the anticipating integral (48) the classical Itô formula remains valid. This simplifies the study of corresponding anticipating stochastic differential equations.

2. After finishing the manuscript we were referred to the paper of Ciesielski, Kerkyacharian and Roynette [1] which contains an exten-

sion of the Riemann–Stieltjes integral to continuous functions from certain Besov spaces by means of their Schauder expansions and a corresponding limit procedure. The application to stochastic integrals with respect to the Wiener process leads to the (extended) Stratonovitch integral. For the case of fractional Brownian motion B^H with $H > 1/2$ and the special integrands f of Hölder exponent greater then $1 - H$ it can be shown that our integral (22) agrees with that of the above authors. This provides the convergence of the Riemann–Stieltjes sums and the corresponding calculation rules which have not been derived in [1]. (Concerning stochastic differential equations with respect to B^H the restriction to such f is natural, since the integral as function of the boundary has this property again.)

6. Postscript

In Part II of the paper our (stochastic) integral is studied in more detail and further extended. In particular, we establish relationships to forward integrals existing in the literature. A pathwise approach to certain (anticipative) SDE with random coefficients by means of the Itô formula is presented. In the special case of adapted processes it agrees with known results.

References

[1] Ciesielski, Z., Kerkyacharian, G., Roynette, R.: Quelques espaces fonctionnels associés á des processus gaussiens, Studia Math. **107**, 171–204 (1993)
[2] Hunt, G.A.: Random Fourier Transforms, Trans. Amer. Math. Soc. **71**, 38–69 (1951)
[3] Itô, K.: Multiple Wiener integral, J. Math. Soc. Japan **3**, 157–169 (1951)
[4] Kahane, J.-P.: Some Random Series of Functions, 2nd edition, Cambridge University Press 1985
[5] Mandelbrot, B.B., van Ness, J.W.: Fractional Brownian Motions, Fractional Noises and Applications, SIAM Review **10**, 422–437 (1968)
[6] Nualart, D.: Non causal stochastic integrals and calculus, in: Stochastic Analysis and Related Topics, H. Korezlioglu and A. S. Ustunel Eds., Lecture Notes in Math. **1316**, 80–129 (1986)
[7] Nikol'ski, S.M. (Ed.): Analysis III – Spaces of differentiable functions, Encyclop. Math. Sciences **26** Springer 1991
[8] Nualart, D., and Pardoux, E.: Stochastic calculus with anticipating integrands, Probab. Th. Rel. Fields **73**, 535–581 (1988)
[9] Pardoux, E.: Application of anticipating stochastic calculus to stochastic differential equations, in: Stochastic Analysis and Related Topics II, H. Korezlioglu and A. S. Ustunel Eds., Lecture Notes in Math. **1444**, 63–105 (1988)
[10] Protter, P.: Stochastic Integration and Differential Equations, Springer 1992.

[11] Samko, S.G., Kilbas, A.A., and Marichev, O.I.: Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach 1993.
[12] Skorohod, A.V.: On a generalization of a stochastic integral, Theory Probab. Appl. **20**, 219–233 (1975)
[13] Winkler, G., Weizsäcker, H.v.: Stochastic Integrals, Vieweg, Advanced Lectures in Mathematics, 1990
[14] Zähle, M.: Fractional differentiation in the self-affine case V – The fractional degree of differentiability, Math. Nachr. **185**, 279–306 (1997)