# INTEGRATION WITH RESPECT TO FRACTAL FUNCTIONS AND STOCHASTIC CALCULUS II

M. Zähle

Mathematical Institute
University of Jena
D–07737 Jena

e-mail: zaehle@minet.uni-jena.de

**Abstract**

The link between fractional and stochastic calculus established in part I of this paper is investigated in more detail. We study a fractional integral operator extending the Lebesgue–Stieltjes integral and introduce a related concept of stochastic integral which is similar to the so–called forward integral in stochastic integration theory. The results are applied to ODE driven by fractal functions and to anticipative SDE whose noise processes possess absolutely continuous generalized covariation processes. A survey on this approach may be found in [21].

*Mathematics Subject Classification*: Primary 60H, Secondary 26A42, 34A

# 0 Introduction

In part I we have introduced an extension of Lebesgue–Stieltjes integrals for integrands and integrators of unbounded variation:

$$\int_a^b f\,dg = (-1)^\alpha \int_a^b D^\alpha_{a+} f_{a+}(x)\, D^{1-\alpha}_{b-} g_{b-}(x)\,dx + f(a+)\big(g(b-) - g(a+)\big) \qquad (1)$$

where $D^\alpha_{a+}$ and $D^{1-\alpha}_{b-}$ are the left– and right–sided Weyl–Marchaud derivatives of orders $\alpha$ and $1-\alpha$, respectively, on the interval $(a,b) \subset \mathbb{R}$ and

$$f_{a+}(x) := 1_{(a,b)}(x)\,\big(f(x) - f(a+)\big)$$
$$g_{b-}(x) := 1_{(a,b)}(x)\,\big(g(x) - g(b-)\big)$$

are the corrected functions continuously vanishing at $a+$ and $b-$, respectively. (The one-sided limits $f(a+)$ and $g(b-)$ are supposed to exist.)
In the present paper we will continue to study this integral and a stochastic extension. Then we will apply our notions to associated deterministic and stochastic differential equations.
*Section 1* is based on the definitions from part I. We recall the notion of the integral and the corresponding function spaces. Theorem 1.2 provides an important continuity property of the integral specified to Besov (or Slobodeckij) spaces of type $W^\alpha_2$.
*Section 2* deals with the associated integral operator

$$f \to \int_0^{(\cdot)} a(f,\varphi)\,dg$$

for fractal parameter functions $\varphi$ from the same spaces and smooth transformation functions $a$. Using Theorem 1.2 we show its continuity and under a stronger smoothness condition on $a$ the local contraction property. The higher–dimensional version is also formulated.
*Section 3* presents the classical change–of–variable formula for the integral in the case where the fractional degree of differentiability of the integrator is greater than 1/2. Here the results of section 2 are used as an approximation tool.
*Section 4* contains the following extension of our integral:

$$\int_a^b f\,dg := \lim_{\varepsilon \searrow 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_a^b f(x)\, \frac{g_{b-}(x+u) - g_{b-}(x)}{u}\,dx\,du\;.$$

In particular, if the corresponding Riemann–Stieltjes sums converge uniformly then the above limit exists and agrees with the Riemann–Stieltjes integral. (In a classical paper of L. C. Young [23] uniform convergence of the sums was proved for functions of finite $p$– and $q$–variations with $1/p + 1/q > 1$.) We further apply this approach to random functions, variable upper bounds of the integrals and uniform convergence in probability. Thus we obtain a modification of a stochastic integral introduced by Russo and Vallois [16] which is more adapted to fractional calculus. For the special case of the Wiener process as integrator we formulate rather general conditions for existence of the integral in terms of Malliavin calculus. Then the integral may be interpreted as the trace corrected Skorohod integral which has been considered by several authors under more restrictive assumptions.

*Section 5* provides a related concept of generalized quadratic variations and covariations of stochastic processes. It is an extension of a notion of Russo and Vallois [17]. We establish relationships to the above fractional calculus and prove the simple version of the Itô formula for such processes. Our approach admits to calculate the bracket of a stochastic integral by means of that of the integrator under rather general conditions (Theorem 5.4).

*Section 6* is concerned with differential equations driven by functions with fractal degree of differentiability greater than 1/2. Working within the Besov–type spaces mentioned above we are able to use the contraction theorem from section 2 in order to prove existence and uniqueness of a local solution. It can be determined by means of Picard's iteration method. In the one–dimensional case (or under certain algebraic conditions on the vector fields) the solutions may be represented as smooth functions of the driving processes (or their iterated integrals). This turns out from the results of section 7. In particular, this provides a pathwise solution procedure for a large class of stochastic processes like fractional Brownian motion with time dependent Hurst exponents.

*Section 7* treats stochastic differential equations driven by processes with absolutely continuous generalized covariation processes. Applying the above stochastic integrals and the simple Itô formula we construct a local pathwise solution for the case of commuting anticipative random vector fields. (The method of reducing to classical (partial) differential equations has already been used by Doss [4] and Sussman [19] in terms of adapted Itô or Stratonovitch integration in a special case.) We also prove uniqueness in the class of all processes satisfying the general Itô transformation formula. Then we extend this approach to the case where the Lie algebra generated by the vector fields is nilpotent of rank $p > 1$. Since we do not require any kind of adaptedness we have to assume that the iterated integrals of order $\leq p$ of the driving processes satisfy the Itô formula. For the algebraic part which is the same as in the adapted Stratonovitch approach we refer to Ikeda and Watanabe [7] and Yamato [22].

# 1 Generalized Stieltjes integrals in $W_2^\alpha$

In part I we defined the integral $\int_a^b f\,dg$ according to (1) provided that $f_{a+} \in I_{a+}^\alpha(L_p)$ , $g(a+)$ exists, $g_{b-} \in I_{b-}^{1-\alpha}(L_q)$ for some $1/p+1/q \le 1$ , $0 \le \alpha \le 1$, where $L_p = L_p(a,b)$ and $I_{a+}^\alpha(L_p)$ denotes the space of functions which are representable as $I_{a+}^\alpha-$ (resp. $I_{b-}^\alpha-$) integral of some $L_p$–function on the interval $(a,b)$. For $\alpha p < 1$ this integral agrees with

$$\int_a^b f\,dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(x)\, D_{b-}^{1-\alpha} g_{b-}(x)\,dx \tag{1'}$$

which is also determined for general $f \in I_{a+}^\alpha(L_p)$ with $\limsup_{x\searrow a} f(x) < \infty$. (This means that in this case we need no correction of $f$ at the left endpoint.
The sets $I^\alpha_{\substack{a+\\(b-)}}(L_p)$ become Banach spaces by the norms

$$\|f\|_{I^\alpha_{\substack{a+\\(b-)}}(L_p)} := \|f\|_{L_p} + \|D^\alpha_{\substack{a+\\(b-)}} f\|_{L_p} \sim \|D^\alpha_{\substack{a+\\(b-)}} f\|_{L_p} \;.$$

For $\alpha p < 1$ the spaces $I_{a+}^\alpha(L_p)$ and $I_{b-}^\alpha(L_p)$ agree up to norm equivalence. Similarly, for any $-\infty \le a < x < y < b \le +\infty$ the restriction of $f \in I_{a+}^\alpha(L_p)$ to the interval $(x,y)$ belongs to $I_{x+}^\alpha\big(L_p(x,y)\big)$ and the continuation of $f \in I_{x+}^\alpha\big(L_p(x,y)\big)$ by zero beyond $(x,y)$ is an element of $I_{a+}^\alpha(L_p)$. (This results from the Hardy–Littlewood inequality, cf. Samko, Kilbas and Marichev [18], chapter 13.) The functions need not be continuous or bounded.

For $\alpha p > 1$ we have embedding in a Hölder space, i.e.,

$$I^\alpha_{\substack{a+\\(b-)}}(L_p) \hookrightarrow H^{\alpha-1/p}$$

and the functions vanish at $a+$ of order $o\big((x-a)^{\alpha-1/p}\big)$ (at $b-$ of order $o\big((b-x)^{\alpha-1/p}\big)$). In this section we will specify to $p=q=2$ and consider the following *Besov– (or Slobodeckij–) type spaces* $W_2^\alpha$ (with modifications) given by the (semi) norms

$$\|f\|_{\widetilde{W}_2^\alpha} := \left( \int_a^b \int_a^b \frac{\big(f(x)-f(y)\big)^2}{|x-y|^{2\alpha+1}}\, dx\, dy \right)^{1/2}$$

$$\|f\|_{W_2^\alpha} := \|f\|_{L_2} + \|f\|_{\widetilde{W}_2^\alpha}$$

$$\|f\|_{W_{2,\infty}^\alpha} := \|f\|_{L_\infty} + \|f\|_{\widetilde{W}_2^\alpha}$$

$$\|f\|_{W_2^\alpha(a+)} := \left( \int_a^b \frac{f(x)^2}{(x-a)^{2\alpha}}\, dx \right)^{1/2} + \|f\|_{\widetilde{W}_2^\alpha}$$

$$\|f\|_{W_2^\alpha(b-)} := \left( \int_a^b \frac{f(x)^2}{(b-x)^{2\alpha}}\, dx \right)^{1/2} + \|f\|_{\widetilde{W}_2^\alpha} \; .$$

When restricting to some subinterval $(x,y) \subset (a,b)$ we will use the notations

$$\widetilde{W}_2^\alpha(x,y) \ , \ W_2^\alpha(x,y) \ , \ W_{2,\infty}^\alpha(x,y) \ , \ W_2^\alpha \underset{(b-)}{(a+)}(x,y).$$

For $(a,b) = \mathbb{R}$ we will write $I^\alpha_{\substack{a+\\(b-)}} = I^\alpha_\pm$ , $\widetilde{W}_2^\alpha(\mathbb{R})$, etc. The above Besov spaces are closely related to $I^\alpha_{\substack{a+\\(b-)}}(L_2)$. Below we will need the following relationships. Suppose $0 < \alpha < 1$.

**1.1 Theorem.**

(i) $\|f\|_{I^\alpha_{\substack{a+\\(b-)}}(L_2)} + \|f\|_{L_\infty} \sim \|f\|_{W_{2,\infty}^\alpha}$ *and* $\|f\|_{I^\alpha_{\substack{a+\\(b-)}}(L_2)} \le \mathrm{const}(\alpha)\, \|1_{(a,b)} f\|_{\widetilde{W}_2^\alpha(\mathbb{R})}$ *if* $0 < \alpha < 1/2$ .

(ii) $W_2^{\alpha+\delta} \underset{(b-)}{(a+)} \hookrightarrow I^\alpha_{\substack{a+\\(b-)}}(L_2)$ , $\delta > 0$ .

(iii) $I^{\alpha+\delta}_{\substack{a+\\(b-)}}(L_2) \hookrightarrow \widetilde{W}_2^\alpha$ , $0 < \delta < 1-\alpha$ *(see Feyel, de La Pradelle [5], Theorem 27).*

(iv) $g \in \widetilde{W}_2^\alpha$ *implies* $g_{y-} \in W_2^\alpha\,(y-)\,(x,y)$ *for any* $x \in [a,b]$ *and Lebesgue almost all* $y \in (x,b)$ *(similarly for* $x+$*).*

*Remark.* The constants in the norm estimates in (i) and (ii) tend to infinity as $\alpha \nearrow 1/2$ and $\delta \searrow 0$, respectively.

*Proof.* **(i)** follows from the norm equivalences

$$\| \cdot \|_{I^\alpha_\pm (L_2(\mathbb{R}))} \sim \| \cdot \|_{W_2^\alpha(\mathbb{R})}$$

(see Mazja and Nagel [13]),

$$\|1_{(a,b)}\, f\|_{I^\alpha_+(L_2(\mathbb{R}))} \sim \|f\|_{I^\alpha_{a+}(L_2\,(a,b))}$$

(cf. [18], 13.3),

$$\|1_{(a,b)}\, f\|_{W^\alpha_{2,\infty}(\mathbb{R})} \sim \|f\|_{W^\alpha_{2,\infty}(a,b)}$$

and the inequality

$$\|1_{(a,b)}\, f\|_{L_2} \leq \text{const}\ \|1_{(a,b)}\, f\|_{\widetilde{W}_2^\alpha(\mathbb{R})}.$$

(The last two facts follow from the definitions.)
**(ii)** $\|f\|_{I^\alpha_{a+}(L_2)}$ up to the summand $\Big( \int\limits_a^b \frac{f(x)^2}{(x-a)^{2\alpha}}\, dx \Big)^{1/2}$ (which is included into the norm of the Besov-type space) and up to some constants does not exceed the limit as $\varepsilon \searrow 0$ of

$$\begin{aligned}
&\left( \int\limits_a^b \left( \int\limits_a^{x-\varepsilon} \frac{f(x)-f(y)}{(y-x)^{2\alpha+1}}\, dy \right)^2 dx \right)^{1/2} \\
&\quad = \left( \int\limits_a^b \left( \int\limits_a^{x-\varepsilon} \frac{f(x)-f(y)}{(y-x)^{\alpha+2\delta}}\, \frac{1}{(y-x)^{1-2\delta}}\, dy \right)^2 dx \right)^{1/2} \\
&\quad \leq \text{const}(\delta) \left( \int\limits_a^b \int\limits_a^{x-\varepsilon} \frac{\big(f(x)-f(y)\big)^2}{(y-x)^{2(\alpha+2\delta)}}\, \frac{1}{(y-x)^{1-2\delta}}\, dy\, dx \right)^{1/2} \\
&\quad = \text{const}(\delta) \left( \int\limits_a^b \int\limits_a^{x-\varepsilon} \frac{\big(f(x)-f(y)\big)^2}{(y-x)^{2(\alpha+\delta)+1}}\, dy\, dx \right)^{1/2} \\
&\quad \leq \text{const}(\delta)\ \|f\|_{\widetilde{W}_2^{\alpha+\delta}}.
\end{aligned}$$

(In the first estimation we have used the Cauchy–Schwarz inequality.) The proof for $b-$ is similar.
**(iv)** Since

$$\int\limits_a^b \int\limits_a^b \frac{\big(g(u)-g(v)\big)^2}{|u-v|^{2\alpha}}\, du\, dv < \infty$$

we infer

$$\int\limits_x^y \int\limits_x^y \frac{\big(g(u)-g(v)\big)^2}{|u-v|^{2\alpha}}\, du\, dv < \infty\ ,$$

i.e. the first part of the assertion, and

$$\int_x^b \int_x^y \frac{\big(g(u)-g(y)\big)^2}{(y-u)^{2\alpha+1}}\, du\, dy < \infty$$

for any $a \le x < y \le b$. The last inequality yields

$$\int_x^y \frac{\big(g(u)-g(y)\big)^2}{(y-u)^{2\alpha+1}}\, du < \infty$$

for almost all $y \in (x,b)$. Consequently, at these $x$ we get

$$\int_x^y \frac{\big(g(u)-g(y)\big)^2}{(y-u)^{2\alpha}}\, du \le (b-a) \int_x^y \frac{\big(g(u)-g(y)\big)^2}{(y-u)^{2\alpha+1}}\, du < \infty$$

which proves the remaining part of the assertion. □

We now turn to *continuity properties* of the integral (1) or (1′) as function of the upper and lower boundaries. They are the key for the main results of this paper.

**1.2 Theorem.** *Suppose* $0 < \alpha < 1/2$ *and* $0 < \beta < 1$.

(i) *If* $f \in I^{\alpha}_{a+}(L_2)$ *and* $g_{b-} \in I^{1-\alpha}_{b-}(L_2)$ *then the integral* $\int_a^b 1_{(x,y)} f\, dg$ *in the sence of* (1′) *exists for any* $a \le x < y \le b$. *Moreover,*

$$\int_a^b 1_{(x,y)} f\, dg = \int_x^y f\, dg$$

*whenever the right–hand side is determined in the sense of* (1′). *(In general, we will use the left–hand side as definition for the right–hand side.)*

(ii) *If* $f$ *and* $g$ *fulfill the conditions of* (i) *and* $f$ *is bounded, i.e.* $f \in W^{\alpha}_{2,\infty}$, *then*

$$\int_a^x f\, dg \quad \textit{and} \quad \int_x^b f\, dg$$

*are continuous functions in* $x \in (a,b)$.

(iii)

$$\max\left( \| \int_a^{(\cdot)} f\, dg \|_{W^{\beta}_{2,\infty}} \,,\ \| \int_{(\cdot)}^b f\, dg \|_{W^{\beta}_{2,\infty}} \right)$$
$$\le \mathrm{const}(\alpha,\beta)\big( \|f\|_{W^{\max(\alpha,\alpha+\beta-1/2)}_{2,\infty}} \, \|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2)} + \|f\|_{L_\infty} \, \|g\|_{\widetilde{W}^{\beta}_2} \big)$$

(iv)

$$\Big\| \int_a^{(\cdot)} f\, dg \Big\|_{W^\beta_{2,\infty}} \le \mathrm{const}\,(\beta)\, \|f\|_{W^\beta_{2,\infty}}\, \|g_{b-}\|_{W^\beta_2(b-)}$$

*provided that* $\beta > 1/2$.

*Remark.* The constants tend to infinity as $\alpha \nearrow 1/2$ in (iii) and as $\beta \searrow 1/2$ in (iv).

*Proof.* **(i)** Since $2\alpha < 1$ the restriction and continuation properties of the space $I^\alpha_{(\cdot)}(L_2)$ mentioned at the beginning of this section imply that $1_{(x,y)} f \in I^\alpha_{a+}\big(L_2(a,b)\big)$ and $f \in I^\alpha_{x+}\big(L_2(x,y)\big)$. Therefore both the integrals are determined. The equality follows from Theorem 2.5 (i) in part I.
**(ii)** Theorem 2.5 (ii) from part I, i.e. additivity of the integral as function of the boundary for continuous $g$, yields

$$\int_a^x f\, dg - \int_a^{x-\delta} f\, dg = \int_{x-\delta}^x f\, dg = \int_a^b 1_{(x-\delta,x)} f\, dg = (-1)^\alpha \int_a^b D^\alpha_{a+}(1_{(x-\delta,x)} f)(y)\, D^{1-\alpha}_{b-}\, g_{b-}(y)\, dy\,.$$

Below we will show that

$$L_2 - \lim_{\delta \searrow 0} D^\alpha_{a+}\,(1_{(x-\delta,x)} f) \equiv 0.$$

Then we obtain from the Cauchy–Schwarz inequality that the last integral tends to zero as $\delta \searrow 0$, i.e. continuity of $\int_a^{(\cdot)} f\, dg$. (The proof for the lower boundary is similar.)
By definition,

$$\Gamma(1-\alpha)\big|D^\alpha_{a+}\,(1_{(x-\delta,x)} f)(y)\big| = \Big| 1_{(x-\delta,x)}(y)\, \frac{f(y)}{(y-a)^\alpha} + \alpha \int_a^y \frac{1_{(x-\delta,x)}(y)\, f(y) - 1_{(x-\delta,x)}(z)\, f(z)}{(y-z)^{\alpha+1}}\, dz \Big|.$$

The first summand of the last sum tends to zero in $L_2$ as $\delta \searrow 0$, since $f$ is bounded and $\alpha < 1/2$. The second summand, say $\alpha S_\delta(y)$, does so by the following arguments: First note that $S_\delta(y) = 0$ if $y < x - \delta$. For $y > x - \delta$ we get

$$S_\delta(y) = 1_{(x-\delta,x)}(y) \Big( \int_a^{x-\delta} \frac{f(y)}{(y-z)^{\alpha+1}}\, dz + \int_{x-\delta}^y \frac{f(y) - f(z)}{(y-z)^{\alpha+1}}\, dz \Big) - 1_{[x,b)}(y) \int_{x-\delta}^x \frac{f(z)}{(y-z)^{\alpha+1}}\, dz\,.$$

The $L_2$–limit as $\delta \searrow 0$ of both the last summands is zero, because $f$ is bounded and $\alpha < 1/2$.
**(iii)** In order to estimate the supremum norm we apply the Cauchy–Schwarz inequality and Theorem 1.1 (i) and obtain for any $x \in (a,b]$

$$\Big|\int_a^x f\,dg\Big| = \Big|\int_a^b 1_{(a,x)} f\,dg\Big| = \Big|\int_a^b D^{\alpha}_{a+}\,(1_{(a,x)}f)(x)\,D^{1-\alpha}_{b-}\,g_{b-}(x)\,dx\Big|$$
$$\le \|1_{(a,x)}f\|_{I^{\alpha}_{a+}(L_2)}\;\|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2)}$$
$$\le \mathrm{const}(\alpha)\,\|1_{(a,x)}f\|_{W^{\alpha}_{2,\infty}}\;\|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2)}$$
$$\le \mathrm{const}(\alpha)\,\|f\|_{W^{\alpha}_{2,\infty}}\;\|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2)}$$

(for different constants here and in the sequel). For the last estimate we have used once more that $\alpha < 1/2$.
We now turn to the $W^{\beta}_2$-seminorm: By the additivity property of the integral we first get

$$\int_a^y f\,dg - \int_a^x f\,dg = \int_x^b 1_{(x,y)} f\,dg = \int_a^b D^{\alpha}_{x+}\,(1_{(x,y)}f_{x+})(z)\,D^{1-\alpha}_{b-}\,g_{b-}(z)\,dz + f(x)\big(g(y)-g(x)\big)$$

at all points $x$ of continuity of $f$. Hence,

$$\left(\int_a^b\int_a^y \frac{\Big(\int_a^y f\,dg - \int_a^x f\,dg\Big)^2}{(y-x)^{2\beta+1}}\,dx\,dy\right)^{\frac12}$$
$$= \left(\int_a^b\int_a^y (y-x)^{-2\beta-1}\Big[\int_x^b D^{\alpha}_{x+}(1_{(x,y)}f_{x+})(z)\,D^{1-\alpha}_{b-}\,g_{b-}(z)\,dz + f(x)\big(g(y)-g(x)\big)\Big]^2 dx\,dy\right)^{\frac12}$$
$$\le \left(\int_a^b\int_a^y (y-x)^{-2\beta-1}\Big(\int_x^b D^{\alpha}_{x+}\,(1_{(x,y)}f_{x+})(z)\,D^{1-\alpha}_{b-}\,g_{b-}(z)\,dz\Big)^2 dx\,dy\right)^{\frac12}$$
$$+ \left(\int_a^b\int_a^y f(x)^2\;\frac{(g(y)-g(x))^2}{(y-x)^{2\beta+1}}\,dx\,dy\right)^{\frac12}$$
$$=: S_1 + S_2$$

The summand $S_2$ does not exceed

$$\|f\|_{L_\infty}\;\|g\|_{\widetilde{W}^{\beta}_2}$$

which corresponds to the second summand in the asserted estimate. For $S_1$ we obtain

$$S_1 \le \left( \int_a^b \int_a^y (y-x)^{-2\beta-1} \, \|1_{(x,y)} f_{x+}\|^2_{I^\alpha_{x+}(L_2(x,b))} \, \|g_{b-}\|^2_{I^{1-\alpha}_{b-}(L_2(x,b))} \, dx \, dy \right)^{\frac{1}{2}} .$$

Regarding

$$\|1_{(x,y)} f_{x+}\|_{I^\alpha_{x+}(L_2(x,b))} \le \mathrm{const}(\alpha) \, \|1_{(x,y)} f\|_{\widetilde{W}^\alpha_2(\mathbb{R})}$$

(cf. Theorem 1.1 (i)) and

$$\|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2(x,b))} \le \|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2)}$$

(which follows from the definitions) we infer

$$S_1 \le \mathrm{const}(\alpha) \left( \int_a^b \int_a^y (y-x)^{-2\beta-1} \, \|1_{(x,y)} f_{x+}\|^2_{\widetilde{W}^\alpha_2(\mathbb{R})} \, dx \, dy \right)^{\frac{1}{2}} \|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2)} .$$

It remains to estimate the integral factor, say $F$, in the last product. Recall that

$$\begin{aligned}
\|1_{(x,y)} f_{x+}\|^2_{\widetilde{W}^\alpha_2(\mathbb{R})} &= \int\int \frac{\big(1_{(x,y)} f_{x+}(u) - 1_{(x,y)} f_{x+}(v)\big)^2}{|u-v|^{2\alpha+1}} \, du \, dv \\
&= \int_x^y \int_x^y \frac{\big(f(u)-f(v)\big)^2}{|u-v|^{2\alpha+1}} \, du \, dv + \int_{-\infty}^x \int_x^y \frac{\big(f(u)-f(x)\big)^2}{(u-v)^{2\alpha+1}} \, du \, dv \\
&\quad + \int_x^y \int_{-\infty}^x \frac{\big(f(v)-f(x)\big)^2}{(v-u)^{2\alpha+1}} \, du \, dv + \int_y^\infty \int_x^y \frac{\big(f(v)-f(x)\big)^2}{(v-u)^{2\alpha+1}} \, du \, dv \\
&= \int_x^y \int_x^y \frac{\big(f(u)-f(v)\big)^2}{|u-v|^{2\alpha+1}} \, du \, dv + 2 \int_{-\infty}^x \int_x^y \frac{\big(f(u)-f(x)\big)^2}{(u-v)^{2\alpha+1}} \, du \, dv \\
&\quad + \int_y^\infty \int_x^y \frac{(f(v)-f(x))^2}{(v-u)^{2\alpha+1}} \, du \, dv \\
&=: S_3 + S_4 + S_5 .
\end{aligned}$$

We next can estimate the summands $S_4$ and $S_5$ by changing the order of integration:

$$S_4 \le \mathrm{const}(\alpha) \int_x^y \frac{\big(f(u)-f(x)\big)^2}{(u-x)^{2\alpha}} \, du,$$

$$S_5 \le \mathrm{const}(\alpha) \int_x^y \frac{\big(f(u)-f(x)\big)^2}{(y-u)^{2\alpha}} \, du.$$

From this we infer

$$\mathrm{const}(\alpha)\, F \le \Bigg( \int_a^b \int_a^y \frac{1}{(y-x)^{2\beta+1}} \int_x^y \int_x^y \frac{\big(f(u)-f(v)\big)^2}{|u-v|^{2\alpha+1}}\, du\, dv\, dx\, dy \Bigg)^{\frac12}$$
$$+ \Bigg( \int_a^b \int_a^y \frac{1}{(y-x)^{2\beta+1}} \int_x^y \frac{\big(f(u)-f(x)\big)^2}{(u-x)^{2\alpha}}\, du\, dx\, dy \Bigg)^{\frac12}$$
$$+ \Bigg( \int_a^b \int_a^y \frac{1}{(y-x)^{2\beta+1}} \int_x^y \frac{(f(u)-f(x))^2}{(y-u)^{2\alpha}}\, du\, dx\, dy \Bigg)^{\frac12}$$
$$=: S_6 + S_7 + S_8\,.$$

Changing again the orders of integration we get

$$S_6 = \Bigg( \int_a^b \int_a^b \frac{\big(f(u)-f(v)\big)^2}{|u-v|^{2\alpha+1}} \int_{u\vee v}^b \int_a^{u\wedge v} \frac{1}{(y-x)^{2\beta+1}}\, dx\, dy\, du\, dv \Bigg)^{\frac12}$$
$$\le \mathrm{const}(\beta) \Bigg( \int_a^b \int_a^b \frac{\big(f(u)-f(v)\big)^2}{|u-v|^{2\alpha+2\beta}}\, dx\, dy \Bigg)^{\frac12}$$
$$= \mathrm{const}(\beta)\, \|f\|_{\widetilde{W}_2^{\alpha+\beta-1/2}}\,.$$

Similarly,

$$S_7 = \Bigg( \int_a^b \int_a^u \frac{\big(f(u)-f(x)\big)^2}{(u-x)^{2\alpha}} \int_u^b \frac{1}{(y-x)^{2\beta+1}}\, dy\, dx\, du \Bigg)^{\frac12}$$
$$\le \mathrm{const}(\beta) \Bigg( \int_a^b \int_a^u \frac{\big(f(u)-f(x)\big)^2}{(u-x)^{2\alpha+2\beta}}\, dx\, du \Bigg)^{\frac12}$$
$$= \mathrm{const}(\beta)\, \|f\|_{\widetilde{W}_2^{\alpha+\beta-1/2}}\,.$$

Finally,

$$S_8 = \Bigg( \int_a^b \int_a^u \big(f(u)-f(x)\big)^2 \int_u^b \frac{1}{(y-x)^{2\beta+1}} \frac{1}{(y-u)^{2\alpha}}\, dy\, dx\, du \Bigg)^{\frac12}.$$

The inner integral equals

$$\int_0^{b-u} \frac{1}{(y+u-x)^{2\beta+1}} \frac{1}{y^{2\alpha}}\, dy = \frac{1}{(u-x)^{2\alpha+2\beta}} \int_0^{\frac{b-u}{u-x}} \frac{1}{(z+1)^{2\beta+1}} \frac{1}{z^{2\alpha}}\, dz$$
$$\le \mathrm{const}(\alpha,\beta)\, \frac{1}{(u-x)^{2\alpha+2\beta}}\,.$$

Hence,

$$S_8 \leq \text{const}(\alpha,\beta)\ \|f\|_{\widetilde{W}_2^{\alpha+\beta-1/2}}\ .$$

This completes the proof for $\int\limits_a^{(\cdot)} f\,dg$. The arguments for $\int\limits_{(\cdot)}^b f\,dg$ are similar.

**(iv)** is a consequence of (iii): Choose there $\alpha$ such that $1-\beta < \alpha < 1/2$ and use Theorem 1.1 (ii) in order to estimate

$$\|g_{b-}\|_{I_{b-}^{1-\alpha}(L_2)} \leq \text{const}(\alpha,\beta)\ \|g_{b-}\|_{W_2^\beta(b-)}\ .$$

□

*Remark.* An analysis of the estimations in [18] and [13] leading to Theorem 1.1 and the proof of the preceding theorem show that const$(\alpha,\beta)$ tends to infinity as $\alpha \nearrow 1/2$, and consequently const$(\beta)$ does so as $\beta \searrow 1/2$.

# 2 An integral operator, continuity and contraction properties

From now on we will frequently use the "time" interval $(0,T)$ instead of $(a,b)$ having in mind applications to related fractal–type (stochastic) differential equations. Let us fix an *integrator* $g$ with $g_{T-} \in W_2^\beta(T-)$ for some $1/2 < \beta < 1$, a "*parameter*" *function* $\varphi \in W_{2,\infty}^\beta$ and a *transformation mapping* $a \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$. By the local Lipschitz property of $a$ for any $f \in W_{2,\infty}^\beta$ the function $a\big(f(\cdot),\varphi(\cdot)\big)$ lies again in this space. Therefore Theorem 1.2 (iv) implies that the non–linear integral operator

$$f \to x_0 + \int\limits_0^{(\cdot)} a(f,\varphi)\,dg$$

for fixed $x_0 \in \mathbb{R}$ acts from $W_{2,\infty}^\beta$ into itself.

Below we will prove continuity of this operator controling the norms (Theorem 2.1). Moreover, in the case when both the partial derivatives of $a$ are locally Lipschitz in the first argument we will infer a certain local contraction property (Theorem 2.2). This provides the key for solving related (anticipative stochastic) differential equations in section 6. As a second result we will derive in section 3 an integral transformation formula using the norm estimates from Theorem 2.1 together with the Lipschitz property of the linear operator

$$g \to x_0 + \int\limits_0^{(\cdot)} a(f,\varphi)\,dg$$

from $W_2^\beta(T-)$ into $L_\infty$.
We first consider the behaviour of the mapping

$$f \to a(f,\varphi)\,.$$

For arbitrary functions $f,h,\varphi \in W_{2,\infty}^\alpha$, where $0<\alpha<1$, denote the closed convex hull of the set $\big(f([0,T]) \cup h([0,T])\big) \times \varphi([0,T])$ in $\mathbb{R}\times\mathbb{R}$ by $K(f,h,\varphi)$. For any compact $K\subset \mathbb{R}\times\mathbb{R}$ denote $\mathcal{L}_0(a,K) := \|\frac{\partial a}{\partial x^1}\|_{L_\infty(K)}$ and let $w_i(a,K;\varepsilon)$, $i=1,2$, be the moduli of continuity of $\frac{\partial a}{\partial x^i}(x^1,x^2)$ on $K$ with respect to the first argument. $\mathcal{L}_i(a,K)$, $i=1,2$ denote the corresponding Lipschitz constants of the partial derivatives if they exist, i.e.,

$$w_i(a,K;\varepsilon) \le \mathcal{L}_i(a,K)\,\varepsilon\,.$$

**2.1 Proposition.**

(i) *For $f,h,\varphi \in W_{2,\infty}^\alpha$ and $a\in C^1$ we have*

$$\|a(f,\varphi)-a(h,\varphi)\|_{L_\infty} \le \mathcal{L}_0\big(a,K(f,h,\varphi)\big)\,\|f-h\|_{L_\infty}$$

*and*

$$\begin{aligned}&\|a(f,\varphi)-a(h,\varphi)\|_{\widetilde{W}_2^\alpha} \le \mathcal{L}_0\big(a,K(f,h,\varphi)\big)\,\|f-h\|_{\widetilde{W}_2^\alpha}\\ &+ w_1\big(a,K(f,h,\varphi);\,\|f-h\|_{L_\infty}\big)\,\min\big(\|f\|_{\widetilde{W}_2^\alpha},\,\|h\|_{\widetilde{W}_2^\alpha}\big)\\ &+ w_2\big(a,K(f,h,\varphi);\,\|f-h\|_{L_\infty}\big)\,\|\varphi\|_{\widetilde{W}_2^\alpha}\end{aligned}$$

(ii) *If $\frac{\partial a}{\partial x^1}$ and $\frac{\partial a}{\partial x^2}$ are locally Lipschitz with respect to the first argument then in* (ii) *$w_i$ may be replaced by $\mathcal{L}_i\big(a,K(f,h,\varphi)\big)\,\|f-h\|_{L_\infty}$, $i=1,2$.*

*Proof.* **(i)** The mean value theorem implies

$$\|a(f,\varphi)-a(h,\varphi)\|_{L_\infty} \le \mathcal{L}_0\big(a,K(f,h,\varphi)\big)\,\|f-h\|_{L_\infty}\,.$$

Futhermore,

$$\begin{aligned}&\Big|a\big(f(s),\varphi(s)\big)-a\big(h(s),\varphi(s)\big)-a\big(f(t),\varphi(t)\big)+a\big(h(t),\varphi(t)\big)\Big|\\ &\le \Big|a\big(f(s),\varphi(s)\big)-a\big(f(t),\varphi(s)\big)-a\big(h(s),\varphi(s)\big)+a\big(h(t),\varphi(s)\big)\Big|\\ &+\Big|a\big(f(t),\varphi(s)\big)-a\big(f(t),\varphi(t)\big)-a\big(h(t),\varphi(s)\big)+a\big(h(t),\varphi(t)\big)\Big|\\ &=: S_1+S_2\,.\end{aligned}$$

By the Leibniz rule we obtain for the first summand:

$$
\begin{aligned}
S_1 = \Bigg| \int_0^1 \frac{\partial a}{\partial x^1} \, \big(\lambda f(s) + (1-\lambda)\, f(t), \varphi(s)\big)\, d\lambda \, \big(f(s) - f(t)\big) \\
- \int_0^1 \frac{\partial a}{\partial x^1} \, \big(\lambda h(s) + (1-\lambda)\, h(t), \varphi(s)\big)\, d\lambda \, \big(h(s) - h(t)\big) \Bigg| \\
\leq \Bigg| \int_0^1 \frac{\partial a}{\partial x^1} \, \big(\lambda f(s) + (1-\lambda)\, f(t), \varphi(s)\big)\, d\lambda \, \big|f(s) - f(t) - h(s) + h(t)\big| \\
+ \Bigg| \int_0^1 \Bigg( \frac{\partial a}{\partial x^1} \, \big(\lambda h(s) + (1-\lambda)\, h(t), \varphi(s)\big) - \frac{\partial a}{\partial x^1} \, \big(\lambda f(s) + (1-\lambda)\, f(t), \varphi(s)\big) \Bigg) \, d\lambda \Bigg| \\
|h(s) - h(t)| \\
\leq \mathcal{L}_0\big(a, K(f,h,\varphi)\big)\, |f(s) - f(t) - h(s) + h(t)| \\
+ w_1\big(a, K(f,h,\varphi);\, \|f-h\|_{L_\infty}\big)\, |h(s) - h(t)| \, .
\end{aligned}
$$

Similarly, the second summand may be estimated by

$$
S_2 \leq w_2\big(a, K(f,h,\varphi); \|f-h\|_{L_\infty}\,\big) |\varphi(s) - \varphi(t)| \, .
$$

Hence,

$$\begin{aligned}
&\big\|a(f,\varphi)-a(h,\varphi)\big\|_{\widetilde{W}_2^\alpha}\\
&=\Bigg(\int\limits_0^T\int\limits_0^T\big(a(f(s),\varphi(s))-a(h(s),\varphi(s))-a(f(t),\varphi(t))+a(h(t),\varphi(t))\big)^2\\
&\qquad\qquad |s-t|^{-(2\alpha+1)}\,ds\,dt\Bigg)^{1/2}\\
&\le \mathcal{L}_0\big(a,K(f,h,\varphi)\big)\,\|f-h\|_{\widetilde{W}_2^\alpha}+w_1\big(a,K(f,h,\varphi);\|f-h\|_{L_\infty}\big)\\
&\quad\Bigg(\int\limits_0^T\int\limits_0^T\frac{\big(h(s)-h(t)\big)^2}{|s-t|^{2\alpha+1}}\,ds\,dt\Bigg)^{1/2}+w_2\big(a,K(f,h,\varphi);\|f-h\|_{L_\infty}\big)\\
&\quad\Bigg(\int\limits_0^T\int\limits_0^T\frac{\big(\varphi(s)-\varphi(t)\big)^2}{|s-t|^{2\alpha+1}}\,ds\,dt\Bigg)^{1/2}\\
&=\mathcal{L}_0\big(a,K(f,h,\varphi)\big)\;\|f-h\|_{\widetilde{W}_2^\alpha}\\
&+w_1\big(a,K(f,h,\varphi);\|f-h\|_{L_\infty}\big)\,\|h\|_{\widetilde{W}_2^\alpha}\\
&+w_2\big(a,K(f,h,\varphi);\|f-h\|_{L_\infty}\big)\,\|\varphi\|_{\widetilde{W}_2^\alpha}\,.
\end{aligned}$$

Since the roles of $f$ and $h$ may be exchanged the proof of (i) is completed.
**(ii)** is an immediate consequence of (i). □

*Remark.* An analysis of the proof shows that the function $\varphi$ may also be chosen vector–valued with coordinate functions $\varphi^1,\dots,\varphi^k$ in $W_{2,\infty}^\alpha$. If $w_2,\dots,w_{k+1}$ are the moduli of continuity of $\frac{\partial a}{\partial y^i}\left(x,y^1,\dots,y^k\right)$, $i=1,\dots,k$, as function in $x$ then $w_2(\cdot)\,\|\varphi\|_{(\cdot)}$ in (i) has to be replaced by

$$\sum_{i=1}^k\; w_{i+1}\big(a,K(f,h,\varphi);\;\|f-h\|_{L_\infty}\big)\,\|\varphi_i\|_{\widetilde{W}_2^\alpha}\;.$$

The following main estimation is an immediate consequence of Theorem 1.2, Proposition 2.1 and Theorem 1.1 (iii).

**2.2 Theorem.**

(i) *Let* $0<\alpha<1/2\,,\,0<\beta<1\,,\,f,h,\varphi\in W_{2,\infty}^{\max(\alpha,\alpha+\beta-1/2)}\,,\,g_{T-}\in$

$I_{T-}^{1-\alpha}(L_2)$ , $g \in \widetilde{W}_2^\beta$ *and* $a \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$. *Then we have*

$$\Big\| \int_0^{(\cdot)} a(f,\varphi)\,dg - \int_0^{(\cdot)} a(h,\varphi)\,dg \Big\|_{W_{2,\infty}^\beta}$$
$$\le \mathrm{const}(\alpha,\beta) \bigg( \Big[ \mathcal{L}_0\big(a, K(f,h,\varphi)\big)\, \|f-h\|_{W_{2,\infty}^{\max(\alpha,\alpha+\beta-1/2)}}$$
$$+ w_1\big(a, K(f,h,\varphi);\, \|f-h\|_{L_\infty}\big) \min\big(\|f\|_{\widetilde{W}_2^{\max(\alpha,\alpha+\beta-1/2)}},\, \|h\|_{\widetilde{W}_2^{\max(\alpha,\alpha+\beta-1/2)}}\big)$$
$$+ w_2\big(a, K(f,h,\varphi);\, \|f-h\|_{L_\infty}\big)\, \|\varphi\|_{\widetilde{W}_2^{\max(\alpha,\alpha+\beta-1/2)}} \Big]$$
$$\|g_{T-}\|_{I_{T-}^{1-\alpha}(L_2)} + \mathcal{L}_0\big(a, K(f,h,\varphi)\big)\, \|f-h\|_{L_\infty}\, \|g\|_{\widetilde{W}_2^\beta} \bigg)$$

(ii)

$$\Big\| \int_{t_0}^{(\cdot)} a(f,\varphi)\,dg - \int_{t_0}^{(\cdot)} a(h,\varphi)\,dg \Big\|_{W_{2,\infty}^\beta(t_0,t)}$$
$$\le \mathrm{const}(\alpha,\beta) \Big[ \mathcal{L}_0\big(a, K(f,h,\varphi)\big)\, \|f-h\|_{W_{2,\infty}^\beta(t_0,t)}$$
$$+ \mathcal{L}_1\big(a, K(f,h,\varphi)\big)\, \|f-h\|_{L_\infty(t_0,t)} \min\big(\|f\|_{\widetilde{W}_2^\beta(t_0,t)},\, \|h\|_{\widetilde{W}_2^\beta(t_0,t)}\big)$$
$$+ \mathcal{L}_2\big(a, K(f,h,\varphi)\big)\, \|f-h\|_{L_\infty(t_0,t)}\, \|\varphi\|_{\widetilde{W}_2^\beta(t_0,t)} \Big]\, \|g_{t-}\|_{W_2^\beta(t_0,t)}$$

*for any* $0 \le t_0 < T$ *and almost all* $t_0 < t \le T$ *provided that* $1/2 < \beta < 1$ , $f,h,\varphi \in W_{2,\infty}^\beta$ , $g \in \widetilde{W}_2^\beta$ , $a \in C^1$ *and the partial derivatives* $\frac{\partial a}{\partial x^1}$ *and* $\frac{\partial a}{\partial x^2}$ *are locally Lipschitz with respect to the first argument.*

As a corollary we now will formulate a local *contraction property* of the integral operator. Denote $W_{2,\infty}^\beta(t_0,t;x_0,1)$ the set of functions $f$ on $(t_0,t)$ with $f(t_0+) = x_0$ and $\|f_{t_0+}\|_{W_{2,\infty}^\beta(t_0,t)} \le 1$.

**2.3 Theorem.** *Let* $x_0, y_0 \in \mathbb{R}$ *and* $\beta, g, a$ *be as in Theorem 2.2* (ii). *Then for any* $t_0 \in (0,T)$ *and* $c > 0$ *there exists some* $t \in (t_0,T)$ *such that for any* $\varphi \in W_{2,\infty}^\beta(t_0,t;y_0,1)$ *the integral operator* $A$ *with*

$$Af := x_0 + \int_{t_0}^{(\cdot)} a(f,\varphi)\,dg$$

*maps* $W_{2,\infty}^\beta(t_0,t;x_0,1)$ *into itself and we have*

$$\|Af - Ah\|_{W_{2,\infty}^\beta(t_0,t)} \le c\|f-h\|_{W_{2,\infty}^\beta(t_0,t)}$$

*for all* $f,h \in W_{2,\infty}^\beta(t_0,t;x_0,1)$.

*Proof.* First note that by Theorem 1.2 (ii) we get

$$\lim_{t\searrow t_0}\Big(x_0+\int_{t_0}^{t} a(f,\varphi)\,dg\Big)=x_0$$

for any $f,\varphi\in W^{\beta}_{2,\infty}(t_0,t)$. Further, Theorem 2.2 (ii) implies for any $f,h,\varphi$ as in the assertion and almost all $t>t_0$

$$\begin{aligned}&\Big\|\int_{t_0}^{(\cdot)} a(f,\varphi)\,dg-\int_{t_0}^{(\cdot)} a(h,\varphi)\,dg\Big\|_{W^{\beta}_{2,\infty}(t_0,t)}\\ &\le \mathrm{const}\|f-h\|_{W^{\beta}_{2,\infty}(t_0,t)}\,\|g_{t-}\|_{W^{\beta}_{2}(t-)(t_0,t)}\,.\end{aligned}$$

Below we will show that

$$\lim_{k\to\infty}\|g_{t_k-}\|_{W^{\beta}_{2}(t_k-)(t_0,t_k)}=0$$

for some sequence $t_k\searrow t_0$. Hence, for large $k$ we get

$$\mathrm{const}\|g_{t_k-}\|_{W^{\beta}_{2}(t_k-)(t_0,t_k)}\le c$$

which leads to the asserted estimation. In order to prove that the norm of the images does not exceed 1 we use Theorem 1.2 (iv) and obtain for almost all $t>t_0$

$$\|Af_{t_0+}\|_{W^{\beta}_{2,\infty}(t_0,t)}\le \mathrm{const}\|a(f,\varphi)\|_{W^{\beta}_{2,\infty}(t_0,t)}\,\|g_{t-}\|_{W^{\beta}_{2}(t-)(t_0,t)}.$$

For $f,\varphi$ as before the first norm on the right–hand side is uniformly bounded and in the second norm we may replace $t$ by the sequence $t_k$ mentioned above in order to make it arbitrarily small.

Thus, it remains to show existence of such a sequence $t_k\searrow t_0$. Recall that

$$\|g_{t-}\|_{W^{\beta}_{2}(t-)(t_0,t)}=\Big(\int_{t_0}^{t}\frac{\big(g(s)-g(t)\big)^2}{(t-s)^{2\beta}}\,ds\Big)^{1/2}+\Big(\int_{t_0}^{t}\int_{t_0}^{t}\frac{\big(g(s)-g(r)\big)^2}{|s-r|^{2\beta+1}}\,ds\,dr\Big)^{1/2}.$$

The second summand goes to zero as $t\searrow t_0$, because it is finite for $(t_0,t)=(0,T)$. In order to prove

$$\lim_{k\to\infty}\int_{t_0}^{t_k}\frac{\big(g(t_k)-g(s)\big)^2}{(t_k-s)^{2\beta}}\,ds=0$$

for some sequence $t_k\searrow t_0$ we consider for $0<\Delta<1$ the auxiliary random variables

$$X_\Delta(u):=\int_{t_0}^{t_0+\Delta u}\frac{\big(g(t_0+\Delta u)-g(s)\big)^2}{(t_0+\Delta u-s)^{2\beta}}\,ds$$

with respect to the normalized Lebesgue measure on the interval $(0, T - t_0)$. Below we will show that the mean value of $X_\Delta$ tends to zero as $\Delta \searrow 0$. Then there exists a sequence $\Delta_k \searrow 0$ such that

$$\lim_{k\to\infty} X_{\Delta_k}(u) = 0$$

for almost all $u$ and we may choose $t_k := t_0 + \Delta_k u$ for any such $u$.
The expectation of $X_\Delta$ equals

$$\frac{1}{T-t_0} \int_0^{T-t_0} \int_{t_0}^{t_0+\Delta u} \frac{\big(g(t_0+\Delta u) - g(s)\big)^2}{(t_0+\Delta u - s)^{2\beta}}\, ds\, du$$

$$= \int_{t_0}^{t_0+\Delta(T-t_0)} \int_{t_0}^{r} \frac{\big(g(r) - g(s)\big)^2}{\Delta(T-t_0)(r-s)^{2\beta}}\, ds\, dr$$

$$\leq \int_{t_0}^{t_0+\Delta(T-t_0)} \int_{t_0}^{r} \frac{\big(g(r) - g(s)\big)^2}{(r-s)^{2\beta+1}}\, ds\, dr$$

which tends to zero as $\Delta \searrow 0$ by the above arguments for the second summand.
□

The following higher–dimensional contraction theorem is a straightforward extension.

**2.4 Theorem.** *The statement of Theorem 2.3 remains valid if* $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^k$, $(g^j) \in \widetilde{W}_2^\beta$, $a_j \in C^1(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^n)$ *with partial derivatives being locally Lipschitz in the first* $n$ *arguments,* $j = 1, \ldots, l$, $\varphi$ *takes values in* $\mathbb{R}^k$ *and* $f$ *and* $h$ *in* $\mathbb{R}^n$ *with coordinate functions as before and the operator is given by*

$$Af := x_0 + \sum_{j=1}^{l} \int_{t_0}^{(\cdot)} a_j(f, \varphi)\, dg^j$$

*with coordinatewise definition of the integrals.*

# 3 Integral transformation formulae

The well–known *change–of–variable* formula for integration of smooth functions remains valid for our fractal–type integral provided the integrator has fractional derivatives of order greater than 1/2. (This condition makes the generalized quadratic variation zero, cf. section 5.)

**3.1 Theorem.** *Let* $0 < \alpha < 1/2$ , $f \in I^{\alpha}_{0+}(L_2)$ *be bounded (i.e.,* $f \in W^{\alpha}_{2,\infty}$*),* $g_{T-} \in I^{1-\alpha}_{T-}(L_2)$ *and*

$$h(t) := h(0) + \int_0^t f\,dg \ , \ t \in (0,T]\,.$$

*Then we get for any* $C^1$*–function* $F(x,t)$ *on* $\mathbb{R} \times [0,T]$ *such that* $\frac{\partial F}{\partial x} \in C^1$ *and for any* $0 \le t_0 < t \le T$ *:*

$$F\big(h(t),t\big) - F\big(h(t_0),t_0\big) = \int_{t_0}^{t} \frac{\partial F}{\partial x}\ \big(h(s),s\big)\ f(s)\ dg(s) + \int_{t_0}^{t} \frac{\partial F}{\partial t}\ \big(h(s),s\big)\,ds\,.$$

*Remark.* Under the more restrictive assumption that $g$ is Hölder continuous of order $\mu > 1/2$ and $f$ is Hölder continuous of order $\lambda > 1-\mu$ we have proved this formula in part I (Theorems 4.3.1 and 4.4.2) for $F \in C^1$ such that $\frac{\partial F}{\partial x}\ \big(h(\cdot),\cdot\big)$ is Hölder continuous of order $\lambda$, in particular for $\frac{\partial F}{\partial x}\ \in C^1$. The smoothness of $\frac{\partial F}{\partial x}$ seems to be the price that we have to pay in order to treat functions of low order of Hölder continuity.

*Proof of Theorem 3.1.* By the usual kernel smoothing procedure we may approximate $g_{T-}$ in the $I^{1-\alpha}_{T-}(L_2)$–norm by smooth functions $g^n$ as $n \to \infty$ (cf. part I). Denote

$$h^n(t) := h(0) + \int_0^t\ f\,dg^n.$$

Then Theorem 1.2 (iii) for $\beta = \alpha$ and Theorem 1.1 (iii) for $\alpha + \delta = 1 - \alpha$ imply

$$\lim_{n\to\infty} \|h^n - h\|_{W^{\alpha}_{2,\infty}} = 0\,.$$

From classical calculus we know that

$$F\big(h^n(t),t\big) - F\big(h^n(t_0),t_0\big) = \int_{t_0}^{t} \frac{\partial F}{\partial x}\big(h^n(s),s\big)\,f(s)\,dg^n(s) + \int_{t_0}^{t} \frac{\partial F}{\partial t}\big(h^n(s),s\big)\,ds.$$

Since $F$ is continuous the left–hand side converges to $F\big(h(t),t\big) - F\big(h(t_0),t_0\big)$ as $n \to \infty$. Similarly,

$$\lim_{n\to\infty} \int_{t_0}^{t} \frac{\partial F}{\partial t}\ \big(h^n(s),s\big)\,ds = \int_{t_0}^{t} \frac{\partial F}{\partial t}\ \big(h(s),s\big)\,ds\,.$$

It remains to show that the first integral tends to

$$\int_{t_0}^{t} \frac{\partial F}{\partial x} \big(h(s), s\big)\, f(s)\, dg(s)\,.$$

The difference may be estimated as follows:

$$\Big| \int_{t_0}^{t} \frac{\partial F}{\partial x} \big(h^n(s), s\big)\, f(s)\, dg^n(s) - \int_{t_0}^{t} \frac{\partial F}{\partial x} \big(h(s), s\big)\, f(s)\, dg(s) \Big|$$

$$\le \Big| \int_{t_0}^{t} \frac{\partial F}{\partial x} \big(h^n(s), s\big)\, f(s)\, dg^n(s) - \int_{t_0}^{t} \frac{\partial F}{\partial x} \big(h(s), s\big)\, f(s)\, dg^n(s) \Big|$$

$$+ \Big| \int_{t_0}^{t} \frac{\partial F}{\partial x} \big(h(s), s\big)\, f(s)\, d\big(g^n(s) - g(s)\big) \Big| \,.$$

The last summand vainishes asymptotically by the same arguments as above for $\|h^n - h\|_{W^{\alpha}_{2,\infty}}$. In order to prove the same property for the first summand we apply the first part of the proof of Theorem 1.2 (iii) and the remark to Proposition 2.1 (ii) to the functions $a(x, y^1, y^2) := \frac{\partial F}{\partial x}\, (x, y^1) \cdot y^2$, if $y^1 \in [0, T]$ , $\varphi^1 :=$ identity, $\varphi^2 := 1_{(t_0,t)} f$ and obtain the upper estimate

$$\text{const}\, \|h^n - h\|_{W^{\alpha}_{2,\infty}}\ \|g^n\|_{I^{1-\alpha}_{T-}(L_2)}\,.$$

Since the last factor is uniformly bounded this tends to zero as $n \to \infty$. □

The higher–dimensional version is again a straight–forward extension:

**3.2 Theorem.** *For $i = 1, \ldots, m$ let $0 < \alpha_i < 1/2$ , $f^i \in I^{\alpha_i}_{0+}(L_2)$ be bounded, $g^i_{T-} \in I^{1-\alpha_i}_{T-}(L_2)$,*

$$h^i(t) := h^i(0) + \int_0^t f^i\, dg^i$$

*and $h := (h^1, \ldots, h^m)$. Then we have for any $C^1$–mapping $F : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ such that $\frac{\partial F}{\partial x^i} \in C^1$, $i = 1, \ldots, m$, and any $0 \le t_o < t \le T$*

$$F\big(h(t), t\big) - F\big(h(t_0), t_0\big) = \sum_{i=1}^{m} \int_{t_0}^{t} \frac{\partial F}{\partial x^i}\big(h(s), s\big)\, f^i(s)\, dg^i(s) + \int_{t_0}^{t} \frac{\partial F}{\partial t}\big(h(s), s\big)\, ds\,.$$

# 4 An extension of the integral and its stochastic version

In part I we have introduced a new notion of stochastic integrals which includes the Itô integral and the trace corrected Skorohod integral for a related type of integrands. Here we will extend these ideas and connect them with the approach of Russo and Vallois [16], [17] for a general anticipative situation.
In order to motivate our notion we start with two relationships for the integral (1) or (1′):

**4.1 Lemma.** *If $f$ and $g$ are as in definitions* (1) *or* (1′) *then the integral may be approximated as follows*

$$\int_a^b f\,dg = \lim_{\varepsilon \searrow 0} \int_a^b I^{\varepsilon}_{a+} f\,dg\,.$$

*Proof.* By definition (1),

$$\lim_{\varepsilon \searrow 0} \int_a^b I^{\varepsilon}_{a+} f\,dg = \lim_{\varepsilon \searrow 0} (-1)^{\alpha} \int_a^b D^{\alpha}_{a+}\ I^{\varepsilon}_{a+}\ f_{a+}(x)\ D^{1-\alpha}_{b-}\ g_{b-}(x)\ dx$$
$$+ f(a+)\ \big(g(b-) - g(a+)\big)\,.$$

Since $D^{\alpha}_{a+}\ I^{\varepsilon}_{a+}\ f_{a+} = I^{\varepsilon}_{a+}\ D^{\alpha}_{a+}\ f_{a+}$ and

$$(L_p) - \lim_{\varepsilon \searrow 0}\ I^{\varepsilon}_{a+}\ D^{\varepsilon}_{a+}\ f_{a+} = D^{\alpha}_{a+}\ f_{a+}$$

(cf. part I), the assertion follows from the Hölder inequality. (The arguments under the conditions of (1′) are similar.) □

The integrals $\int\limits_a^b I^{\varepsilon}_{a+} f\,dg$ are also determined for $f$ and $g$ of slightly lower order of differentiability. In order to treat the limit case we introduce the function spaces

$$I^{\beta-}_{\substack{a+\\(b-)}}(L_p) := \bigcap_{\alpha<\beta} I^{\alpha}_{\substack{a+\\(b-)}}(L_p)$$

and similarly $W^{\beta-}_{2,\infty}$, etc. The following representation leads to a relationship between our approach and that of Russo and Vallois.

**4.2 Lemma.** *Suppose $\frac{1}{p} + \frac{1}{q} \le 1$ and $\varepsilon > 0$.*

(i)

$$\int_a^b I_{a+}^{\varepsilon} f\, dg = \frac{1}{\Gamma(\varepsilon)} \lim_{\delta \searrow 0} \int_\delta^\infty u^{\varepsilon-1} \int_a^b f(s) \frac{g_{b-}(s+u) - g_{b-}(u)}{u}\, ds\, du$$

*provided that* $f \in I_{a+}^{\alpha-\varepsilon}(L_p)$ , $g_{b-} \in I_{b-}^{1-\alpha}(L_q)$ *with* $\alpha p \neq 1$.

(ii) *The equation in* (i) *holds true if* $f \in I_{a+}^{\beta-}(L_p)$ , $g_{b-} \in I_{b-}^{(1-\beta)-}(L_q)$ *for some* $0 < \beta < 1$.

*Proof.* **(i)** In order to transform the right–hand side into the left–hand side we will use the special composition formula

$$I_{a+}^{\alpha-\varepsilon}\, D_{a+}^{\alpha-\varepsilon}\, f = f$$

together with the integration–by–part rule

$$\int_a^b I_{a+}^{\alpha-\varepsilon}\, \varphi(s)\, \psi(s)\, ds = (-1)^{\alpha-\varepsilon} \int_a^b \varphi(s)\, I_{b-}^{\alpha-\varepsilon}\, \psi(s)\, ds$$

for $\varphi := D_{a+}^{\alpha-\varepsilon} f$ and $\psi(s) := g_{b-}(s+u) - g_{b-}(s)$ , $u > 0$ (cf. part I). Then we obtain

$$\begin{aligned}\int_a^b f(s) \big(g_{b-}(s+u) - g_{b-}(s)\big)\, ds &= \int_a^b I_{a+}^{\alpha-\varepsilon}\, D_{a+}^{\alpha-\varepsilon} f(s) \big(g_{b-}(s+u) - g_{b-}(s)\big)\, ds \\ &= (-1)^{\alpha-\varepsilon} \int_a^b D_{a+}^{\alpha-\varepsilon} f(s) \big[I_{b-}^{\alpha-\varepsilon}\, g_{b-}(s+u) - I_{b-}^{\alpha-\varepsilon}\, g_{b-}(s)\big]\, ds \\ &=: \Phi(u)\end{aligned}$$

Note that the right–hand side of the assertion agrees with

$$\lim_{\delta \searrow 0} \int_\delta^\infty \frac{1}{\Gamma(\varepsilon)}\, u^{\varepsilon-2}\, \Phi(u)\, du\,.$$

According to Fubini's theorem we have

$$\int_\delta^\infty \frac{1}{\Gamma(\varepsilon)}\, u^{\varepsilon-2}\, \Phi(u)\, du = (-1)^{\alpha-\varepsilon} \int_a^b D_{a+}^{\alpha-\varepsilon} f(s) \int_\delta^\infty \frac{1}{\Gamma(\varepsilon)} \frac{I_{b-}^{\alpha-\varepsilon}\, g_{b-}(s+u) - I_{b-}^{\alpha-\varepsilon}\, g_{b-}(s)}{u^{1-\varepsilon+1}}\, du\, ds$$

and the inner integral converges in $L_q$ as $\delta \downarrow 0$ to

$$(-1)^{\varepsilon}\, D_{b-}^{1-\varepsilon}\, I_{b-}^{\alpha-\varepsilon}\, g_{b-}(s) = (-1)^{\varepsilon}\, D_{b-}^{1-\alpha}\, g_{b-}(s)\,.$$

Using $D_{a+}^{\alpha-\varepsilon} f_{a+} = D_{a+}^{\alpha} I_{a+}^{\varepsilon} f_{a+}$ and the Hölder inequality we infer

$$\lim_{\delta\downarrow 0} \frac{1}{\Gamma(\varepsilon)} \int_{\delta}^{\infty} u^{\varepsilon-2}\, \Phi(u)\, du = (-1)^{\alpha} \int_{a}^{b} D_{a+}^{\alpha} I_{a+}^{\varepsilon} f(s)\, D_{b-}^{1-\alpha} g_{b-}(s)\, ds$$
$$= \int_{a}^{b} I_{a+}^{\varepsilon} f(s)\, dg(s)$$

according to definition ($1'$) if $\alpha p < 1$ and definition (1) if $\alpha p > 1$, since in the latter case $I_{a+}^{\varepsilon} f(a+) = 0$.

**(ii)** is a consequence of (i) if we replace there $\alpha$ by $\beta + \Delta\varepsilon$ for some $0 < \Delta < 1$ such that $(\beta + \Delta\varepsilon)\, p \neq 1$. □

*Remark.* In the stochastic calculus below we will consider $\beta = 1/2$ and $p = q = 2$. An analysis of the last proof and Theorem 1.1 (i) show that in this case the upper boundary $b$ in the integrals may be replaced by any $t \in (a, b]$ and convergence holds uniformly in $t$.

Lemmas 4.1 and 4.2 suggest the following *extension* of our integral (1) or ($1'$) (regarding that $\Gamma(\varepsilon)$ is equivalent to $\varepsilon^{-1}$ as $\varepsilon \searrow 0$):

**Definition.**

$$\int_{a}^{b} f\, dg := \lim_{\varepsilon\searrow 0} \varepsilon \lim_{\delta\searrow 0} \int_{\delta}^{1} u^{\varepsilon-1} \int_{a}^{b} f(s)\, \frac{g_{b-}(s+u) - g_{b-}(s)}{u}\, ds\, du \tag{2}$$

whenever the right–hand side exists.

Note, that the kernel $\varepsilon\, u^{\varepsilon-1}$ acts as the $\delta$–function as $\varepsilon \searrow 0$. If

$$\lim_{\Delta\searrow 0} \int_{a}^{b} f(s)\, \frac{g_{b-}(s+\Delta) - g_{b-}(s)}{\Delta}\, ds$$

exists then the integral (2) is determined and agrees with this limit. The notion of general Riemann–Stieltjes integral in the sense of uniform convergence with respect to a "random" starting point may be considered as special case:

**4.3 Proposition.** *If the Riemann–Stieltjes sums*

$$S_{\Delta}(x) := \sum_{k=0}^{\left[\frac{b-a}{\Delta}\right]-1} f(x + k\Delta)\, \Big(g\big(x + (k+1)\Delta\big) - g(x + k\Delta)\Big)$$

*converge uniformly in* $x \in (a, a+\Delta)$ *to a limit* $(R-S)\int\limits_a^b f\,dg$ *as* $\Delta \searrow 0$ *then we have*

$$(R-S)\int\limits_a^b f\,dg = \lim_{\Delta\searrow 0}\int\limits_a^b f(s)\,\frac{g_{b-}(s+\Delta)-g_{b-}(s)}{\Delta}\,ds$$

*provided that the Lebesgue integrals on the right–hand side exist.*

*Remark.* In Young [23] such a convergence has been proved for $f$ and $g$ being of finite $p$– and $q$–variations, where $\frac{1}{p}+\frac{1}{q} > 1$. We conjecture that this may be extended to a notion of generalized $p$–variation (see section 5 for $p=2$) and convergence in the sense of (2).

*Proof of Proposition 4.3.* By the assumption we get

$$\lim_{\Delta\searrow 0}\frac{1}{\Delta}\int\limits_0^\Delta S_\Delta(x)\,dx = (R-S)\int\limits_a^b f\,dg$$

and in the definition of $S_\Delta$ the function $g$ may be replaced by $g_{b-}$ without changing the limit. Using

$$\begin{aligned}\int\limits_0^\Delta S_\Delta(x)\,dx &= \sum_k \int\limits_0^\Delta f(x+k\Delta)\Big(g_{b-}\big(x+(k+1)\Delta\big)-g_{b-}(x+k\Delta)\Big)\,dx\\ &= \sum_k \int\limits_{k\Delta}^{(k+1)\Delta} f(s)\big(g_{b-}(s+\Delta)-g_{b-}(s)\big)\,ds\\ &= \int\limits_a^b f(s)\big(g_{b-}(s+\Delta)-g_{b-}(s)\big)\,ds\end{aligned}$$

we obtain the assertion. □

We now will apply this approach to stochastic processes on the interval $[0,T]$. The integrals as functions of the upper boundary $t$ will again be considered as random processes and for limits we will use *uniform convergence* in *probability* for $t \in [0,T]$ briefly $\lim\limits_{(ucp)}$, or convergence in the mean square denoted by l.i.m. Suppose that $Y$ is a càglàd (left continuous with right limits) process and $Z$ a càdlàg (right continuous with left limits) process on $[0,T]$.

**Definition.**

$$\Bigg(\int\limits_{0+}^{t-} Y\,dZ = \Bigg)\int\limits_0^t Y\,dZ := \lim_{\varepsilon\searrow 0}\ \varepsilon\int\limits_0^1 u^{\varepsilon-1}\int\limits_0^t Y(s)\,\frac{Z_{t-}(s+u)-Z_{t-}(s)}{u}\,ds\,du \quad (3)$$

whenever the right–hand side is determined, where lim stands for one of the stochastic limits above and $\int\limits_0^1$ for $\lim\limits_{\delta\searrow 0}\int\limits_\delta^1$ with probability 1.

If the right–hand side exists for $\lim\limits_{(ucp)}$ as well as for l.i.m. then the resulting processes are stochastically equivalent.

We immediately obtain the following sample path property of our integral (3) if it is determined via (*ucp*): The process

$$X(t) := \int\limits_0^{t+} Y dZ$$

is càdlàg and $X(t) - X(t-) = Y(t)\big(Z(t) - Z(t-)\big)$. Moreover, continuity of $Z$ implies that of $X$.

*Remark.* Russo and Vallois use (up to replacing $Y(s)$ by $Y(s+)$) the following definition of stochastic (forward) integral:

$$\int\limits_0^t Y dZ := \lim_{\substack{u\searrow 0\\(ucp)}} \int\limits_0^t Y(s)\, \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u}\, ds \tag{3'}$$

i.e., a limit procedure without averaging. Because of convergence in probability, in general, the integrals in (3′) and (3) seem to be different. However, if we assume in (3′) convergence in the mean squared (for $L_2$–processes on $\Omega \times [0,T]$) or convergence with probability 1 then this implies the corresponding convergence in (3) and the integrals agree. Because of the decomposition theorem for *semimartingales* this may be applied to the classical situation of *Itô calculus*:

**4.4 Proposition.** *If $Z$ is a semimartingale and $Y$ an adapted càglàd process then the integrals in* (3) *and* (3′) *are determined in the* (*ucp*)*–sense and agree with the usual Itô–integral*

$$(I)\int\limits_{0+}^{t-} Y dZ\,.$$

A proof for (3′) and the Itô–integral is given in [17], Proposition 1.1. Since it uses stopping times and convergence in the mean square it may immediately be adapted to the "average" convergence in (3). □

As a further special case we will consider the following extension of the *trace corrected Skorohod integral* known from *Malliavin calculus*. Let $Z := W$ be the Wiener process and $X \in \mathbb{L}^{1,2}$. Then $1_{(0,t)}X$ is Skorohod integrable for any $t \in$

$(0,1)$. As usual we write $\delta(1_{(0,t)}X)$ for the Skorohod integral. Moreover, the corresponding *derivative* $D_s X(t)$ satisfies $\mathbb{E}\int_0^1\int_0^1 D_s X(t)^2\,ds\,dt < \infty$. (For more details see, e.g. [14], [15].) The next result extends Theorem 5.3.7 in part I as well as the corresponding results in [16].

**4.5 Theorem.** *If for* $\varepsilon \to 0$,

$$\varepsilon \int_0^1 u^{\varepsilon-1} \int_0^{(\cdot)} \frac{1}{u} \int_s^{s+u} D_r X_s\,dr\,ds\,du$$

*converges in* $L_2$ *on* $\Omega \times [0,1]$ *to a random process denoted by* $Tr(DX)^+$ *then*

$$\varepsilon \int_0^1 u^{\varepsilon-1} \int_0^{(\cdot)} X(s)\ \frac{W(s+u)-W(s)}{u}\ ds\,du$$

*converges in the same sense to*

$$\int_0^{(\cdot)} X dW = \delta(1_{(0,\cdot)}X) + Tr(DX)^+(\cdot)\,.$$

*Proof.* (*This short version was suggested by a referee.*)
Using the rules of Malliavin calculus we get

$$\frac{1}{u}\int_0^t X(s)\big(W(s+u)-W(s)\big)\,ds = \frac{1}{u}\int_0^t\int_s^{s+u} X_s\,dW_r\,ds + \frac{1}{u}\int_0^t\int_s^{s+u} D_r X_s\,dr\,ds\,.$$

The averages of the second summand converge to $Tr(DX)^+(t)$ by assumption. According to the Fubini theorem for the Skorohod integral (which follows from the definition of $\delta$ as dual operator to the derivative) we obtain for the first summand the expression

$$\int_0^t \frac{1}{u}\int_{0\vee(r-u)}^{(t-u)\wedge r} X(s)\,ds\ dW(r)\,.$$

Straigtforward calculations show that for $u \searrow 0$ the process under this Skorohod integral is asymptotically $\mathbb{L}^{1,2}$-equivalent to

$$Y_u(r) := \frac{1}{u}\int_{r-u\vee 0}^{r} X(s)ds$$

which converges in $\mathbb{L}^{1,2}$ as $u \searrow 0$ to $X(r)$. This implies the $L_2$–convergence of the above integral to the Skorohod integral of $X$. The corresponding convergence of the averages w.r.t. $u$ is a consequence. □

*Remark.* Note that the type of averaging in the limit does not play a role as long as it is assumed for the trace $Tr(D)^+$. In a forthcoming paper we will show how other kinds of averaging known from analysis fit into our model.
The resulting integral is a forward integral which, in general, does not agree with the Stratonovich integral, where the symmetric trace has to be added to the Skorohod integral instead of $Tr(DX)^+$.
Under the sharper trace condition

$$\operatorname*{l.i.m.}_{u\searrow 0}\ \mathbb{E}\int_0^1 \left(\frac{1}{u}\int_s^{s+u} D_r X(s)\,dr - D_{s+}X(s)\right)^2 ds = 0$$

one obtains the result of Russo and Vallois [16]

$$\operatorname*{l.i.m.}_{u\searrow 0}\int_0^t X(s)\ \frac{W(s+u)-W(s)}{u}\ du = \delta(1_{(0,t)}X) + \int_0^t D_{s+}X(s)\,ds\,.$$

The pointwise trace condition in Asch and Potthoff [2] also implies that of Theorem 4.5 and the right–hand side agrees with their definition of the integral.

# 5 Processes with generalized quadratic variation and Itô formula

Let $\mathcal{D}$ be the set of càdlàg functions on $[0,T]$.

**Definition.** A process $Z \in \mathcal{D}$ admits a *generalized quadratic variation process (bracket)* $[Z](t)$ if

$$[Z](t) := \lim_{\substack{\varepsilon\searrow 0\\(ucp)}}\ \varepsilon\int_0^1 u^{\varepsilon-1}\int_0^t \frac{1}{u}\ \big(Z_{t-}(s+u)-Z_{t-}(s)\big)^2\,ds\,du + \big(Z(t)-Z(t-)\big)^2 \quad (4)$$

exists. The covariation process $[Y,Z]$ of $Y, Z \in \mathcal{D}$ with generalized brackets is defined similarly, where $(\dots)^2$ is replaced by the corresponding product.

*Remark.* $[Y,Z](t)$ if exists is a càdlàg process of bounded variation and $[Z](t)$ is non–decreasing. Moreover, $[Y,Z](t) - [Y,Z](t-) = \big(Y(t)-Y(t-)\big)\big(Z(t) - Z(t-)\big)$. (Russo and Vallois [17] use again limits without averaging.) Note that semimartingales or, more generally, Dirichlet processes are special examples and the bracket agrees with that defined in the corresponding theory. In particular, we may consider fractional Brownian motion $B^H$ with Hurst exponent $1/2 < H < 1$ as a Dirichlet process with $[B^H] \equiv 0$.

An immediate consequence of the definition of $[Z]$ is the following.

**5.1 Proposition.** *Any $Z \in \mathcal{D}$ admitting a generalized bracket lies with probability 1 in the function space $W_{2,\infty}^{1/2-}$.*

(From now on we will often omit the phrase " with probability 1" if it is clear from the context.)

Most of the properties obtained by Russo and Vallois for their generalized bracket remain valid for our notion:

**5.2 Proposition.** (Cf. [17], Proposition 1.2)
*Let $Y, Z \in \mathcal{D}$ be processes with generalized bracket $[Y, Z]$ and define $[Y, Z]^\varepsilon$ by*

$$[Y,Z]^\varepsilon(t) = \varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t \frac{1}{u} \,\big(Y_{t-}(s+u) - Y_{t-}(s)\big)\big(Z_{t-}(s+u) - Z_{t-}(s)\big)\, ds\, du \\ + \big(Y(t) - Y(t-)\big)\big(Z(t) - Z(t-)\big)\,.$$

*Then we have*

$$\lim_{\substack{\varepsilon \searrow 0 \\ (ucp)}} \int_0^t X\, d[Y,Z]^\varepsilon = \int_0^t X\, d[Y,Z]$$

*for any càglàd process $X$ on $[0,T]$.*

**5.3 Proposition.** (Cf. [17], Proposition 2.1)
*Let $Y, Z$ be continuous processes with generalized brackets $[Y], [Z]$ and $[Y, Z]$ and $F, G$ be random $C^1$–functions on $\mathbb{R}$. Then $F(Y)$ and $G(Z)$ admit a mutual bracket given by*

$$\big[F(Y), G(Z)\big] = \int_0^{(\cdot)} F'(Y)\, G'(Z)\, d[Y,Z]\,.$$

**5.4 Theorem.** *Suppose that $Z$ is continuous, admits a generalized bracket and $X$ is representable as*

$$X(t) = \int_0^t Y\, d\, Z$$

*for some càglàd process $Y$.*

(i) *If $Y \in W_{2,\infty}^{\beta}$ with $\beta > 1/2$ then $X$ admits a generalized bracket given by*

$$[X](t) = \int_0^t Y^2\, d[Z]\,.$$

(ii) *The equation in* (i) *remains valid if* $Y \in W^{1/2-}_{2,\infty}$ *and the stochastic integral converges in the strong sense that*

$$\lim_{\varepsilon \searrow 0} \sup_{\gamma > 0} \frac{1}{\Gamma(\gamma)} \, \|X^\varepsilon - X\|^2_{W^{1/2-\gamma/2}_{2,\infty}} = 0$$

*where*

$$X^\varepsilon(t) = \frac{1}{\Gamma(\varepsilon)} \lim_{\delta \searrow 0} \int_\delta^\infty u^{\varepsilon-1} \int_0^t Y(s) \, \frac{1}{u} \big(Z_{t-}(s+u) - Z_{t-}(s)\big) \, ds \, du$$

*Remark.* Since $Z \in W^{1/2-}_{2,\infty}$ the stochastic integral in (i) may pathwise be interpreted in the sense of definition (1). Recall that $Y$ is here continuous.

*Proof.* **(i)** The equalities

$$X(s+u) - X(s) = \int_s^{s+u} Y \, dZ \quad \text{and} \quad X(t-) - X(s) = \int_s^t Y \, dZ,$$

imply $X_{t-}(s+u) - X_{t-}(s) = \int_s^{s+u} Y \, dZ_{t-}$.

Hence,

$$\varepsilon \int_0^1 u^{\varepsilon-2} \int_0^t \big(X_{t-}(s+u) - X_{t-}(s)\big)^2 \, ds \, du = \varepsilon \int_0^1 u^{\varepsilon-2} \int_0^t \Big( \int_s^{s+u} Y dZ_{t-} \Big)^2 \, ds \, du$$

$$= \varepsilon \int_0^1 u^{\varepsilon-1} \, \frac{1}{u} \int_0^t \Big( \int_s^{s+u} Y(s) \, dZ_{t-}(r) + \int_s^{s+u} (Y(r) - Y(s)) \, dZ_{t-}(r) \Big)^2 \, ds \, du \, .$$

Below we will show that

$$\lim_{u \searrow 0} \frac{1}{u} \int_0^t \Big( \int_s^{s+u} \big(Y(r) - Y(s)\big) \, dZ_{t-}(r) \Big)^2 \, ds = 0$$

uniformly in $t$ with probability 1. Therefore the limit as $\varepsilon \searrow 0$ of the above expression equals

$$\lim_{\substack{\varepsilon \searrow 0 \\ (ucp)}} \varepsilon \int_0^1 u^{\varepsilon-2} \int_0^t Y(s)^2 \big(Z_{t-}(s+u) - Z_{t-}(s)\big)^2 \, ds \, du = \int_0^t Y^2 \, d[Z]$$

according to Proposition 5.2 which leads to the asserted equality.
In order to complete the proof of (i) we choose an arbitrary $\alpha \in (1/2, \beta)$ and estimate using the Cauchy–Schwarz inequality and Theorem 1.1 as follows:

$$
\begin{aligned}
&\left(\frac{1}{u}\int_0^t \Big(\int_s^{s+u} (Y(r)-Y(s))\, dZ_{t-}(r)\Big)^2 ds\right)^{1/2} \\
&= \left(\frac{1}{u}\int_0^{(t-u)_+} \Big(\int_s^{s+u} D^{\alpha}_{s+}\, Y_{s+}(r)\; D^{1-\alpha}_{(s+u)-}\, Z_{(s+u)-}(r)\, dr\Big)^2 ds \right.\\
&\qquad \left. + \frac{1}{u}\int_{(t-u)_+}^{t} \Big(\int_s^{t} D^{\alpha}_{s+}\, Y_{s+}(r)\; D^{1-\alpha}_{t-}\, Z_{t-}(r)\, dr\Big)^2 ds\right)^{1/2} \\
&\le \text{const} \left(\frac{1}{u}\int_0^t \int_s^{(s+u)\wedge t} (D^{\alpha}_{s+}\, Y_{s+}(r))^2\, dr\, ds\right)^{1/2} \\
&\le \text{const} \left(\frac{1}{u}\int_0^t \int_s^{(s+u)\wedge t} \frac{(Y(r)-Y(s))^2}{(r-s)^{2\beta}}\, dr\, ds\right)^{1/2} \\
&\qquad + \text{const} \left(\frac{1}{u}\int_0^t \int_s^{(s+u)\wedge t}\int_s^{(s+u)\wedge t} \frac{(Y(r)-Y(v))^2}{|r-v|^{2\beta+1}}\, dr\, dv\, ds\right)^{1/2} \\
&=: \text{const}\ S_1 + \text{const}\ S_2\,.
\end{aligned}
$$

For $S_1$ we get

$$
S_1^2 = \int_0^t \int_s^{(s+u)\wedge t} \frac{\big(Y(r)-Y(s)\big)^2}{u(r-s)^{2\beta}}\, dr\, ds \le \int_0^T \int_s^{(s+u)\wedge T} \frac{\big(Y(r)-Y(s)\big)^2}{(r-s)^{2\beta+1}}\, dr\, ds
$$

which goes to zero as $u \searrow 0$ since

$$
\int_0^T \int_0^T \frac{\big(Y(r)-Y(s)\big)^2}{|r-s|^{2\beta+1}}\, dr\, ds < \infty\ .
$$

Similarly,

$$
S_2^2 = \frac{1}{u}\int_0^t \int_s^{(s+u)\wedge t}\int_s^{(s+u)\wedge t} \frac{\big(Y(r)-Y(v)\big)^2}{|r-v|^{2\beta+1}}\, dr\, dv\, ds \le 2\int_0^T \int_v^{(v+u)\wedge T} \frac{\big(Y(r)-Y(v)\big)^2}{|r-v|^{2\beta+1}}\, dr\, dv
$$

which is twice the estimator of $S_1^2$.

**(ii)** In view of Lemma 4.2 (i) we have

$$X^\varepsilon(t) = \int_0^t I_{0+}^\varepsilon \, Y \, dZ \, .$$

The processes $I_{0+}^\varepsilon Y$ and $Z$ satisfy the conditions of (i) (cf. Theorem 1.1). Consequently,

$$[X^\varepsilon](t) = \int_0^t (I_{0+}^\varepsilon Y)^2 \; d[Z] \, .$$

Since $Y$ is bounded and left continuous and $d[Z]$ is a bounded measure, the last integrals converge uniformly in $t$ to $\int_0^t Y^2 \, d[Z]$ with probability 1. On the other hand, by definition of the bracket and continuity of $Z$,

$$[X^\varepsilon](t) = \lim_{\substack{\gamma \searrow 0 \\ (ucp)}} \frac{1}{\Gamma(\gamma)} \, \|X^\varepsilon\|^2_{\widetilde{W}_2^{1/2-\gamma/2}(0,t)} \, .$$

The uniform convergence of $\frac{1}{\Gamma(\gamma)}\|X^\varepsilon - X\|^2_{\widetilde{W}_2^{1/2-\gamma/2}}$ to zero as $\varepsilon \searrow 0$ admits to change the limits in $\varepsilon$ and $\gamma$, so that

$$\begin{aligned} \lim_{\substack{\varepsilon \searrow 0 \\ (ucp)}} [X^\varepsilon] &= \lim_{\substack{\gamma \searrow 0 \\ ucp)}} \frac{1}{\Gamma(\gamma)} \lim_{\substack{\varepsilon \searrow 0 \\ (ucp)}} \|X^\varepsilon\|^2_{\widetilde{W}_2^{1/2-\gamma/2}(0,\cdot)} \\ &= \lim_{\substack{\gamma \searrow 0 \\ (ucp)}} \frac{1}{\Gamma(\gamma)} \, \|X\|^2_{\widetilde{W}_2^{1/2-\gamma/2}(0,\cdot)} \, . \end{aligned}$$

Again by definition and continuity of $X$ the right–hand side agrees with $[X]$. (We always use that (*ucp*) is equivalent to uniform convergence with probability 1 of subsequences.) Combining this with the above convergence of the left–hand side we obtain the assertion. □

Similarly as in [17] one can prove the following *Itô–type formula* for the change of variables. (The ideas go back to Föllmer [6], where the Taylor formula and Riemann sums approximation is used. Russo's and Vallois' [17] approach is based on the integral representation of the remainder in the Taylor expansion. With slight modifications this may be transformed to average convergence.)

**5.5 Theorem.** *Let $Z$ be a continuous process with generalized bracket $[Z]$. Then we get for any random $C^1$–function $F(x,t)$ on $\mathbb{R} \times [0,T]$ with continuous $\frac{\partial^2 F}{\partial x^2}$ and*

$0 \leq t_0\,,\, t \leq T$

$$F\big(Z(t),t\big) - F\big(Z(t_0),t_0\big) = \int_{t_0}^{t} \frac{\partial F}{\partial x}\,\big(Z(s),s\big)\,dZ(s) + \int_{t_0}^{t} \frac{\partial F}{\partial t}\,\big(Z(s),s\big)\,ds$$
$$+ \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 F}{\partial x^2}\,\big(Z(s),s\big)\,d[Z](s)$$

*and the stochastic integral is determined by* (*ucp*)*.*

More generally, as an analogue to the classical situation in Itô calculus for semimartingales we define the following for $Z$ and $F$ as in Theorem 5.5:
The process $X$ with

$$X(t) = \int_{0}^{t} Y\,dZ$$

for some cáglád process $Y$ satisfies the *general Itô formula* if

$$F\big(X(t),t\big) - F\big(X(t_0),t_0\big) = \int_{t_0}^{t} \frac{\partial F}{\partial x}\,\big(X(s),s\big)\,Y(s)\,dZ(s) + \int_{t_0}^{t} \frac{\partial F}{\partial t}\,\big(X(s),s\big)\,ds$$
$$+ \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 F}{\partial x^2}\,\big(X(s),s\big)\,Y(s)^2\,d[Z](s). \tag{5}$$

*Remark.*

1) Under the conditions of Theorem 5.4 we obtain from Theorem 5.5

$$F\big(X(t),t\big) - F\big(X(t_0),t\big) = \int_{t_0}^{t} \frac{\partial F}{\partial x}\,\big(X(s),s\big)\,dX(s) + \int_{t_0}^{t} \frac{\partial F}{\partial t}\,\big(X(s),s\big)\,ds$$
$$+ \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 F}{\partial x^2}\,\big(X(s),s\big)\,Y(s)^2\,d[Z](s)\,.$$

Thus, the validity of the general Itô formula in this case is equivalent to

$$\int_{t_0}^{t} \frac{\partial F}{\partial x}\,\big(X(s),s\big)\,dX(s) = \int_{t_0}^{t} \frac{\partial F}{\partial x}\,\big(X(s),s\big)\,Y(s)\,dZ(s)\,.$$

In the case of deterministic $F$ it is well–known for continuous semimartingales and for anticipative integrals $X$ with respect to $Z = W$ under certain conditions. (For the latter cf. [2], [1] and their references.)

2) If $Y \in W_{2,\infty}^{1/2-}$ and $Z \in W_2^{\beta}(T-)$ for some $\beta > 1/2$ we may apply the results of section 3 to the sample paths of the processes and Theorem 3.1 provides the general Itô formula with $[Z] \equiv 0$.

3) We will adopt the general Itô formula (5) and its multidimensional extension as a main calculation rule in the stochastic calculus based on definitions (3) and (4). In particular, solutions of SDE will be seeked only in the class of processes satisfying this condition.

The higher–dimensional version of Theorem 5.5 reads as follows.

**5.6 Theorem.** *Let $Z = (Z^1, \dots, Z^p)$ be a continuous $\mathbb{R}^p$–valued process admitting generalized brackets $[Z^j, Z^k]$ and $F$ be a random element of $C^1(\mathbb{R}^p \times [0,T], \mathbb{R}^n)$ with continuous partial derivatives $\frac{\partial^2 F}{\partial x^j\, \partial x^k}$, $1 \le j,k \le p$. Then we have*

$$F\big(Z(t),t\big) - F\big(Z(t_0),t_0\big) = \sum_{j=1}^{p} \int_{t_0}^{t} \frac{\partial F}{\partial x^j}\big(Z(s),s\big)\, dZ^j(s) + \int_{t_0}^{t} \frac{\partial F}{\partial t}\big(Z(s),s\big)\, ds$$
$$+ \frac{1}{2} \sum_{j,k=1}^{p} \int_{t_0}^{t} \frac{\partial^2 F}{\partial x^j\, \partial x^k}\big(Z(s),s\big)\, d[Z^j, Z^k](s)\,.$$

(In the Taylor expansion techniques behind the stochastic integrals can only be determined in the sense of (*ucp*) of the sum of the approximating integrals.)

# 6 Differential equations driven by fractal functions of order greater than 1/2

In [9] the one–dimensional differential equation

$$\begin{aligned} dx(t) &= a\big(x(t),t\big)\, dz(t) + b\big(x(t),t\big)\, dt \\ x(t_0) &= x_0 \end{aligned}$$

is investigated, where the fractal noise function $z$ is of Hölder continuity of order greater than 1/2 and $a$ and $b$ possess certain smoothness properties. This equation becomes precise via integration:

$$x(t) = x_0 + \int_{t_0}^{t} a\big(x(s),s\big)\, dz(s) + \int_{t_0}^{t} b\big(x(s),s\big)\, ds$$

where the first integral may be interpreted in the sence of (1), but also as general Riemann–Stieltjes integral. The unique Hölder continuous (of order greater than 1/2) local solution is explicitly represented as

$$x(t) = h\big(y(t) + z(t), t\big)$$

for some functions $h$ and $y$ satisfying certain classical differential equations. An extension of this approach may be found in the thesis [8]. In the paper [12] of Lyons existence and uniqueness of the local solution of the following more general equation in $\mathbb{R}^n$ is proved by means of Picard's iteration method:

$$x(t) = x_0 + \sum_{j=1}^{n} \int_0^t a_j\big(x(s)\big)\, dz^j(s)$$

where the $a_j$ are $n$–dimensional vector fields in $\mathbb{R}^n$ with certain smoothness properties and $z(t) = \big(z^1(t), \ldots, z^n(t)\big)$ is a continuous vector function with finite $p$–variation, $p < 2$. (Note that such functions are special elements of the Hölder space $H_p^{1/p}$.) There an extension of a result of Young [23] concerning the general Riemann–Stieltjes integral of such functions is used and the solution is again of finite $p$–variation. Note that Lyons does not prove a contraction principle.
Here we will work in the somewhat different space $W_{2,\infty}^{\beta}$ , $\beta > 1/2$, which will ensure the contraction principle. We consider the differential equation

$$\begin{aligned} dx(t) &= \sum_{j=1}^{l} a_j\big(x(t), \varphi(t)\big)\, dz^j(t) \\ x(t_0) &= x_0 \end{aligned} \tag{6}$$

for some initial values $t_0 \in (0, T)$ , $x_0 \in \mathbb{R}$ and a *driving function* $z = (z^1, \ldots, z^l)$ with $z^j \in W_{2,\infty}^{\beta}$. The additional *parameter function* $\varphi$ takes values in $\mathbb{R}^k$ with coordinate functions in $W_{2,\infty}^{\beta}$ and the $a_j$ are $\mathbb{R}^n$–valued $C^1$*–vector fields* on $\mathbb{R}^n \times \mathbb{R}^k$ such that all $n + k$ *partial derivatives* are *locally Lipschitz* in the first $n$ variables. We seek a local solution of (6) in the space $W_{2,\infty}^{\beta}(t_1, t_2)$ for certain $t_0 \in (t_1, t_2) \subset (0, T)$. Again we interpret (6) via integration according to (1). For $t > t_0$ we can apply the contraction theorem 2.4 and Picard's iteration method. Similarly, for $t < t_0$ equation (6) means

$$x(t) = x_0 - \sum_{j=1}^{l} \int_t^{t_0} a_j\big(x(s), \varphi(s)\big)\, dz^j(s)\,.$$

The integral $\int_t^{t_0} a_j\big(x(s), s\big)\, dz^j(s) =: \int_t^{t_0} f\, dg$ may be understood as

$$\int_a^b 1_{(t,t_0)} f\, dg = (-1)^{\alpha} \int_a^l D_{a+}^{\alpha}\, (1_{(t,t_0)} f)(s)\, D_{b-}^{1-\alpha}\, g_{b-}(s)\, ds$$

for any $0 \le a \le t$ and almost all $b > t_0$ (where $g_{b-} \in W_2^\beta(b-)$), $1/2 < 1-\alpha < \beta$. According to Theorem 3.1 in part I it agrees with the "backward" integral

$$(-1)^\alpha \int_a^b D_{b-}^\alpha \left(1_{(t,t_0)} f\right)(s)\, D_{a+}^{1-\alpha}\, g_{a+}(s)\, ds$$

for almost all a (where $g_{a+} \in W_2^\beta(a+)(a,b)$). Via time reversion with respect to the starting moment $t_0$ the roles of $a+$ and $b-$ in this integral may be exchanged, so that the contraction theorem is also applicable to the backward integral. This leads to the following.

**6.1 Theorem.** *Under the above conditions there exists some interval $(t_1, t_2)$ containig $t_0$ such that equation (6) has a solution in $W_{2,\infty}^\beta(t_1,t_2)$. It may be determined by means of Picard's iteration method which is contractive. The solution is unique on the maximal interval of definition.*

*Remark.* In particular, we may choose $k = 1$, $l = m+1$, $\varphi(t) = z_{m+1}(t) = t$, $a_{m+1} = b$ and obtain the equation in $\mathbb{R}^n$

$$dx(t) = \sum_{j=1}^m a_j\big(x(t),t\big)\, dz^j(t) + b\big(x(t),t\big)\, dt$$
$$x(t_0) = x_0\,.$$

Here the first sum may be interpreted as a fractal noise term added to a classical non–autonomous ordinary differential equation. If the vector fields $a_1, \dots, a_m$ commute then the solution may again be represented as a smooth function of the noise $z = (z^1, \dots, z^m)$ and a smooth function $y$ solving an ODE (cf. section 7).

In general, the *solution* $x = x_\varphi$ of (6) *depends continuously on the parameter function* $\varphi \in W_{2,\infty}^\beta(t_1,t_2)$ with $\varphi(t_0) = y_0$.

**6.2 Theorem.** *Let $y_0 \in \mathbb{R}^k$ and $0 < C < 1 \le K$ be given. Suppose that the conditions of Theorem 6.1 are fulfilled and the vector fields $a_j$ have locally Lipschitz partial derivatives. Then there is a sufficiently small interval $(t_1, t_2)$ containing $t_0$ such that for any two parameter functions $\varphi, \psi$ with $\varphi(t_0) = \psi(t_0) = y_0$ and $\|\varphi - y_0\|_{W_{2,\infty}^\beta(t_1,t_2)} \le K$, $\|\psi - y_0\|_{W_{2,\infty}^\beta(t_1,t_2)} \le K$ the solutions $x_\varphi$ and $x_\psi$ of (6) exist on $(t_1, t_2)$ and satisfy*

$$\|x_\varphi - x_\psi\|_{W_{2,\infty}^\beta(t_0,t)} < C\, \|\varphi - \psi\|_{W_{2,\infty}^\beta(t_0,t)}\,.$$

*Proof.* Without loss of generality we may assume that $K = 1$. (Otherwise consider $a(x, Ky)$ instead of $a(x, y)$.) Replacing in Theorem 2.4 the functions $(f, h, \varphi)$

by $(\varphi, \psi, f)$ we obtain for a sufficiently small interval $(t_1, t_2) \ni t_0$ the operator

$$A_f\, \varphi := x_0 + \sum_{j=1}^{l} \int_{t_0}^{(\cdot)} a_j\big(f(s), \varphi(s)\big)\, dz^j(s)$$

for arbitrary $f \in W^\beta_{2,\infty}(t_1, t_2)$ with $f(t_0) = x_0$ and $\|f - x_0\|_{W^\beta_{2,\infty}(t_1,t_2)} \leq 1$ satisfies

$$\|A_f \phi - A_f \psi\|_{W^\beta_{2,\infty}(t_1,t_2)} \leq \frac{C}{1+C}\, \|\varphi - \psi\|_{W^\beta_{2,\infty}(t_1,t_2)}\,.$$

By the original version of Theorem 2.4 (for $(f, h, \psi)$) we may choose $t_1, t_2$ so that

$$\|A_f \varphi - A_h \phi\|_{W^\beta_{2,\infty}(t_1,t_2)} \leq \frac{C}{1+C}\, \|f - h\|_{W^\beta_{2,\infty}(t_1,t_2)}$$

for arbitrary $h \in W^\beta_{2,\infty}(t_1, t_2)$ with $h(t_0) = x_0$ and $\|h - x_0\|_{W^\beta_{2,\infty}(t_1,t_2)} \leq 1$.
Picard's iteration method provides the solutions $x_\varphi$ and $x_\psi$ and the corresponding approximations in the $n$–th step $x^{(n)}_\varphi$ , $x^{(n)}_\psi$ which fulfill the conditions on $f$ and $h$ as above. For brevity we will omit in the resulting norm estimations the subscription $W^\beta_{2,\infty}(t_1, t_2)$ and note that

$$\|x_\varphi - x_\psi\| = \lim_{n\to\infty} \|x^{(n)}_\varphi - x^{(n)}_\psi\|\,.$$

The above arguments yield

$$\begin{aligned}
\|x^{(n)}_\varphi - x^{(n)}_\psi\| &= \|A_{x^{(n-1)}_\varphi}\, \varphi - A_{x^{(n-1)}_\psi}\, \psi\| \\
&\leq \|A_{x^{(n-1)}_\varphi}\, \varphi - A_{x^{(n-1)}_\psi}\, \varphi\| + \|A_{x^{(n-1)}_\psi}\, \varphi - A_{x^{(n-1)}_\psi}\, \psi\| \\
&\leq \frac{C}{1+C}\, \|x^{(n-1)}_\varphi - x^{(n-1)}_\psi\| + \frac{C}{1+C}\, \|\varphi - \psi\|\,.
\end{aligned}$$

By induction we get

$$\begin{aligned}
\|x^{(n)}_\varphi - x^{(n)}_\psi\| &\leq \sum_{i=1}^{n-1} \left(\frac{C}{1+C}\right)^i \|\varphi - \psi\| \leq \sum_{i=0}^{\infty} \left(\frac{C}{1+C}\right)^i \|\varphi - \psi\| \\
&= C\, \|\varphi - \psi\|\,.
\end{aligned}$$

Hence,

$$\|x_\varphi - x_\psi\| \leq C\, \|\varphi - \psi\|\,.$$

□

# 7 Stochastic differential equations with fractal noise

## 7.1 The case of commuting vector fields

We now return to the random case and apply the results of sections 4 and 5 to SDE.
Let $Z^1, \dots, Z^m$ be one–dimensional continuous random processes on $[0,T]$ with generalized covariation processes of the form

$$[Z^j, Z^k](t) = \int_0^t q^{jk}(s)\, ds$$

for some continuous random functions $q^{jk}$. Denote $Z(t_0) =: Z_0$. We consider the *stochastic differential equation* in $\mathbb{R}^n$

$$\begin{aligned} dX(t) &= \sum_{j=1}^{m} a_j\big(X(t),t\big)\, dZ^j(t) + b\big(X(t),t\big)\, dt \\ X(t_0) &= X_0 \end{aligned} \tag{7}$$

for certain *random vector fields* $a_1, \dots, a_m, b$ and an arbitrary *random initial vector* $X_0$.

**Definition.** A *solution* of (7) is a continuous random process $X = (X^1, \dots, X^n)$ admitting generalized covariation processes $[X^j, X^k]$ which satisfies the multidimensional version of the Itô formula (5) with respect to its coordinatewise integral representation

$$X(t) = X_0 + \sum_{j=1}^{m} \int_{t_0}^{t} a_j\big(X(s),s\big)\, dZ^j(s) + \int_{t_0}^{t} b\big(X(s),s\big)\, ds$$

where the noise term is defined by convergence of the sum of the approximating integrals with respect to the $Z^j$ via (3).

We first consider the case $m = 1$, i.e., $[Z](t) = \int_0^t q(s)\, ds$ and

$$\begin{aligned} dX(t) &= a\big(X(t),t\big)\, dt + b\big(X(t),t\big)\, dt \\ X(t_0) &= X_0\,. \end{aligned} \tag{8}$$

Here we will construct a pathwise solution basing on the Itô formula. For the random vector fields $a$ and $b$ we assume the following conditions (with probability 1):

(C1) $a \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$, all partial derivatives are locally Lipschitz in $x \in \mathbb{R}^n$

(C2) $b \in C(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$, $b(x,t)$ is locally Lipschitz in $x \in \mathbb{R}^n$.

We first consider pathwise the *auxiliary partial differential* equation on $\mathbb{R}^n \times \mathbb{R} \times [0,T]$

$$\begin{aligned} \frac{\partial h}{\partial z}(y,z,t) &= a\big(h(y,z,t),t\big) \\ h(Y_0, Z_0, t_0) &= X_0 \end{aligned} \tag{9}$$

where $Y_0$ is a arbitrary random vector in $\mathbb{R}^n$. Picard's iteration method provides w.p.1 a (non–unique) local solution $h \in C^1$ in a neighborhood of $(Y_0, Z_0, t_0)$ with partial derivatives being Lipschitz in $y$ and

$$\det\left(\frac{\partial h}{\partial y}(y,z,t)\right) \neq 0\,.$$

Moreover,

$$\frac{\partial^2 h}{\partial z^2}(y,z,t) = \sum_{j=1}^n \frac{\partial a}{\partial x^j}\big(h(y,z,t),t\big)\, a^j\big(h(y,z,t),t\big)$$

We seek the solution $X$ of (8) in the form

$$X(t) = h\big(Y(t),\, Z(t),\, t\big) \tag{10}$$

for a random $C^1$–process $Y$ in $\mathbb{R}^n$ with $Y(t_0) = Y_0$ to be determined (in dependence on the choice of $h$). Applying the Itô formula to $F(z,t) := h\big(Y(t),z,t\big)$ we obtain

$$\begin{aligned} dX(t) &= \frac{\partial h}{\partial z}\big(Y(t),Z(t),t\big)\, dZ(t) + \sum_{k=1}^n \frac{\partial h}{\partial y^k}\big(Y(t),Z(t),t\big)\, \dot{Y}^k(t)\, dt \\ &\quad + \frac{\partial h}{\partial t}\big(Y(t),Z(t),t\big)\, dt + \frac{1}{2}\,\frac{\partial^2 h}{\partial z^2}\big(Y(t),Z(t),t\big)\, q(t)\, dt \\ &= a\big(X(t),t\big)\, dZ(t) + \sum_{k=1}^n \frac{\partial h}{\partial y^k}\big(Y(t),Z(t),t\big)\, \dot{Y}^k(t)\, dt \\ &\quad + \frac{\partial h}{\partial t}\big(Y(t),Z(t),t\big)\, dt + \frac{1}{2}\sum_{j=1}^n \frac{\partial a}{\partial x^j}\big(h(Y(t),Z(t),t),t\big)\, a^j\big(h(Y(t),Z(t),t),t\big)\, q(t)\, dt. \end{aligned}$$

Comparing the coefficients we are lead to the following ordinary differential equation for $Y$ (in matrix representation):

$$\begin{aligned}\dot{Y}(t) = \left(\frac{\partial h}{\partial y}\big(Y(t), Z(t), t\big)\right)^{-1} \Big[ & b\big(h(Y(t), Z(t), t), t\big) - \frac{\partial h}{\partial t}\big(Y(t), Z(t), t\big) \\ & - \frac{1}{2} q(t) \left(\frac{\partial a}{\partial x}\big(h(Y(t), Z(t), t), t\big)\right) a\big(h(Y(t), Z(t), t), t\big)\Big] \\ Y(t_0) = Y_0 \, . & \end{aligned} \tag{11}$$

The unique local solution $Y(t)$ may be determined via Picard's iteration method, which is contractive.

**7.1.1 Theorem.** (Existence and uniqueness)

(i) *Under the conditions (C1) and (C2) any representation of the form* (10) *with $h$ fulfilling* (9) *as above and $Y$ determined by* (11) *provides a solution of the SDE* (8).

(ii) *If $X$ is an arbitrary solution of* (8) *in the sense of the above definition then it agrees with any of the representations in* (i) *on the common interval of definition.*

*Proof.* **(i)** follows immediately from the above construction applying the Itô formula (s. Theorem 5.5).
**(ii)** Take $h\big(Y(t), Z(t), t\big)$ as in (i) and let $X(t)$ be another local solution of (8). The mapping

$$(y, z, t) \to \big(h(y, z, t), z, t\big)$$

is invertible in a neighborhood of $(Y_0, Z_0, t_0)$. Let $\big(u(x, z, t), z, t\big)$ be the inverse mapping, i.e.,

$$u\big(h(y, z, t), z, t\big) = y \, .$$

Then we get the matrix equality

$$\left(\frac{\partial u}{\partial x}(x, z, t)\right) = \left(\frac{\partial h}{\partial y}\big(u(x, z, t), z, t\big)\right)^{-1}$$

and furthermore,

$$
\begin{aligned}
\frac{\partial u}{\partial z}(x,z,t) &= -\sum_{k=1}^{n} \frac{\partial u}{\partial x^k}\,(x,z,t)\,a^k(x,t) \\
\frac{\partial u}{\partial t}(x,z,t) &= -\sum_{k=1}^{n} \frac{\partial u}{\partial x^k}\,(x,z,t)\,\frac{\partial h^k}{\partial t}\,\big(u(x,z,t),z,t\big) \\
\frac{\partial^2 u}{\partial z^2}(x,z,t) &= -\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial x^j \partial x^k}\,(x,z,t)\,a^j(x,t)\,a^k(x,t)
\end{aligned}
$$

$$
-\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z\,\partial x^k}\,(x,z,t)\,a^k(x,t) - \sum_{j,k=1}^{n} \frac{\partial u}{\partial x^k}\,(x,z,t)\,\frac{\partial a^k}{\partial x^j}\,(x,t)\,a^j(x,t)\,.
$$

We now will apply the higher–dimensional version of the Itô formula (5) to the function $u(x,z,t)$ and the process $\big(X(t),Z(t)\big)$ given in the integral representation

$$
\begin{aligned}
X(t) &= X_0 + \int_{t_0}^{t} a\big(X(s),s\big)\,dZ(s) + \int_{t_0}^{t} b\big(X(s),s\big)\,ds \\
Z(t) &= Z_0 + \int_{t_0}^{t} dZ(s)
\end{aligned}
$$

in a neighborhood of $t_0$. Since $u(X_0, Z_0, t_0) = Y_0$ this yields

$$
\begin{aligned}
u\big(X(t), Z(t), t\big) - Y_0 = &\int_{t_0}^{t} \frac{\partial u}{\partial z}\,\big(X(s), Z(s), s\big)\, dZ(s) \\
&+ \sum_{k=1}^{n} \int_{t_0}^{t} \frac{\partial u}{\partial x^k}\,\big(X(s), Z(s), s\big)\, a^k\big(X(s), s\big)\, dZ(s) \\
&+ \frac{1}{2} \sum_{j,k=1}^{n} \int_{t_0}^{t} \frac{\partial^2 u}{\partial x^j\, \partial x^k}\,\big(X(s), Z(s), s\big)\, a^j\big(X(s), s\big)\, a^k\big(X(s), s\big)\, q(s)\, ds \\
&+ \frac{1}{2} \sum_{k=1}^{n} \int_{t_0}^{t} \frac{\partial^2 u}{\partial z\, \partial x^k}\,\big(X(s), Z(s), s\big)\, a^k\big(X(s), s\big)\, q(s)\, ds \\
&+ \frac{1}{2} \sum_{k=1}^{n} \int_{t_0}^{t} \frac{\partial^2 u}{\partial z^2}\,\big(X(s), Z(s), s\big)\, q(s)\, ds + \int_{t_0}^{t} \frac{\partial u}{\partial t}\,\big(X(s), Z(s), s\big)\, ds \\
&+ \sum_{k=1}^{n} \int_{t_0}^{t} \frac{\partial u}{\partial x^k}\,\big(X(s), Z(s), s\big)\, b^k\big(X(s), s\big)\, ds\,.
\end{aligned}
$$

Substituting the above expressions for the derivatives we get

$$
\begin{aligned}
u\big(X(t), Z(t), t\big) = Y_0 + \sum_{k=1}^{n} \int_{t_0}^{t} \Bigg[ &\frac{\partial u}{\partial x^k}\,\big(X(s), Z(s), s\big)\, b^k\big(X(s), s\big) \\
- \frac{\partial u}{\partial x^k}\,\big(X(s), Z(s), s\big)\, \frac{\partial h^k}{\partial t}\, &\Big(u\big(X(s), Z(s), s\big), s\Big) - \frac{1}{2} \sum_{j=1}^{k} \frac{\partial u}{\partial x^k}\,\big(X(s), Z(s), s\big) \\
&\frac{\partial a^k}{\partial x^j}\,\big(X(s), s\big)\, a^j\big(X(s), s\big)\, q(s) \Bigg]\, ds
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\frac{d}{dt}\, u\big(X(t), Z(t), t\big) = \left( \frac{\partial u}{\partial x}\big(X(t), Z(t), t\big) \right) \Bigg[ &b\big(X(t), t\big) - \frac{\partial h}{\partial t}\, \Big(u\big(X(t), Z(t), t\big)\Big) - \\
&\frac{1}{2}\, q(t) \left( \frac{\partial a}{\partial x}\, \big(X(t), t\big) \right) a\big(X(t), t\big) \Bigg]\,.
\end{aligned}
$$

Regarding the above matrix equality for $\left(\frac{\partial u}{\partial x}\right)$ we infer that $\widetilde{Y}(t) := u\bigl(X(t), Z(t), t\bigr)$ satisfies the ODE (11):

$$\begin{aligned}
\dot{\widetilde{Y}}(t) &= \left(\frac{\partial h}{\partial y}\,(\widetilde{Y}(t), Z(t), t)\right)^{-1} \left[b\Bigl(h\bigl(\widetilde{Y}(t), Z(t), t\bigr), t\Bigr) - \frac{\partial h}{\partial t}\,(\widetilde{Y}(t), Z(t), t)\right. \\
&\quad \left. - \frac{1}{2}\,q(t)\,\frac{\partial a}{\partial x}\,\Bigl(h\bigl(\widetilde{Y}(t), Z(t), t\bigr), t\Bigr)\,a\bigl(h(\widetilde{Y}(t), Z(t), t), t\bigr)\right] \\
\widetilde{Y}(t_0) &= Y_0\,.
\end{aligned}$$

By uniqueness of the solution of (11) we obtain $\widetilde{Y}(t) = Y(t)$, i.e., $u\bigl(X(t), Z(t), t\bigr) = Y(t)$, which implies $X(t) = h\bigl(Y(t), Z(t), t\bigr)$ in a neighborhood of $t_0$. Since $t_0$ may be replaced by an arbitrary $t$ with $\det\left(\frac{\partial h}{\partial y}\,(Y(t), Z(t), t)\right) \neq 0$ the assertion follows. □

**7.1.2 Theorem.** *For $Z \in W^{\beta}_{2,\infty}$ with $\beta > 1/2$ the results of Theorem 7.1.1 remain valid if the condition* (C1) *is weakened to $a \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$ w.p.1.*

*Proof.* First note that in this case the quadratic variation process vanishes, i.e., $q \equiv 0$. We may use Theorem 3.2 for the corresponding Itô formula. The rest is the same as in the previous proof. □

We next will consider two *special cases*:

1) If we choose for $h$ the unique solution of

$$\frac{\partial h}{\partial z}\,(y, z, t) = a\bigl(h(y, z, t), t\bigr)$$

$$h(y, Z_0, t) = y$$

then we obtain the approach suggested by Doss [4] and Sussman [19] for the special case of non–random time autonomous vector fields within classical Itô calculus.

2) If $\det\left(\frac{\partial a}{\partial x}(X_0, t_0)\right) \neq 0$ we can take

$$h(y, z, t) = \widetilde{h}(y + z, t)$$

with

$$\frac{\partial \widetilde{h}}{\partial z}(z, t) = a\bigl(\widetilde{h}(z, t), t\bigr)$$

$$\widetilde{h}(Z_0, t_0) = X_0$$

For $n = 1$ this has been treated in [9].

We now turn to the case $m > 1$ with *commuting random vector fields* $a_j$. Here we replace condition (C1) by the following:

(C1′) $a_j \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$, all partial derivatives are locally Lipschitz in $x$ and $[\![a_j, a_k]\!] \equiv 0$, $1 \leq j,k \leq m$, where $[\![\cdot,\cdot]\!]$ denotes the Lie bracket of the vector fields with respect to the argument $x \in \mathbb{R}^n$ which is determined almost everywhere.

The PDE (9) takes now the higher–dimensional form on $\mathbb{R}^n \times \mathbb{R}^m \times [0,T]$

$$\begin{aligned} \frac{\partial h}{\partial z^j}(y,z,t) &= a_j\big(h(y,z,t),t\big)\,,\ j = 1,\dots,m \\ h(Y_0, Z_0, t_0) &= X_0\,. \end{aligned} \tag{9′}$$

The commutativity property of the vector fields $a_j$ guarantees the existence of a $C^1$–solution in a neightborhood of $(Y_0, Z_0, t_0)$ with

$$\det\left(\frac{\partial h}{\partial y}(y,z,t)\right) \neq 0$$

and $\frac{\partial h}{\partial t}(y,z,t)$ being Lipschitz in $y$. The remaining procedure is the same as before: The local solution of (7) is unique and may be represented in the form

$$X(t) = h\big(Y(t), Z(t), t\big) \tag{10′}$$

where the random $C^1$–process $Y$ is uniquely determined by the following extension of the ODE (9):

$$\begin{aligned} \dot{Y}(t) &= \left(\frac{\partial h}{\partial y}\big(Y(t),Z(t),t\big)\right)^{-1}\Bigg[b\Big(h\big(Y(t),Z(t),t\big),t\Big) - \frac{\partial h}{\partial t}\big(Y(t),Z(t),t\big) \\ &\quad - \frac{1}{2}\sum_{j,k=1}^{m} q_{jk}(t)\left(\frac{\partial a_j}{\partial x}\Big(h\big(Y(t),Z(t),t\big),t\Big)\right) a_k\Big(h\big(Y(t),Z(t),t\big),t\Big)\Bigg] \\ Y(t_0) &= Y_0\,. \end{aligned} \tag{11′}$$

*Remark.* The problem of existence of a global solution may be reduced to the same question for the differential equations (9′) and (11′), i.e., to corresponding growth conditions on the vector fields.

## 7.2 The case of nilpotent Lie algebras generated by the vector fields

In [22] Yamato extended the Doss approach for Itô or Stratonovitch SDE to the case where the Lie algebra generated by the vector fields $a_1, \dots, a_m$ is nilpotent of order $p > 1$. This method also works for our anticipative integrals with time dependent random vector fields provided that the iterated integrals of the processes $Z^1, \ldots, Z^m$ up to order $p$ satisfy the Itô formula for the functions under consideration. For simplicity we will demonstrate the main ideas on the case $p = 2$ (see also Ikeda and Watanabe [7], chapter III, 2, example 2.2 for Stratonovitch SDE). We suppose the following:

(C1″) $a_j \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$, $\frac{\partial a_j}{\partial t}(x,t)$ and $\frac{\partial^2 a_j}{\partial x^i \partial x^k}(x,t)$ are locally Lipschitz in $x$ and $[\![a_j, [\![a_l, a_m]\!]]\!] \equiv 0$, $j, l, m = 1, \ldots m$, $i, k = 1, \ldots n$, where the Lie bracket is again taken with respect to the space argument $x$.

The condition (C2) is as before.

If the vector fields $a_j$ do not commute then the PDE (9′) is not integrable. Therefore the space $\mathbb{R}^m$ will be enlarged to $\mathbb{R}^M$ with $M = m + m(m-1)/2$. The elements are denoted by

$$\widetilde{z} = (z^1, \ldots, z^m, z^{12}, \ldots, z^{ik}, \ldots, z^{m-1m})$$

with ordered pairs $i < k$. (9′) is replaced by a system of PDE on $\mathbb{R} \times \mathbb{R}^M \times [0,T]$ with values in $\mathbb{R}^n$:

$$\begin{aligned} \frac{\partial h}{\partial z^j}(y, \widetilde{z}, t) &= a_j\big(h(y,\widetilde{z},t),t\big) - \sum_{i=1}^{j-1} z^i [\![a_i, a_j]\!]\big(h(y,\widetilde{z},t),t\big), \; j = 1, \ldots, m \\ \frac{\partial h}{\partial z^{jk}}(y, \widetilde{z}, t) &= [\![a_j,\, a_k]\!]\big(h(y,\widetilde{z},t),t\big), \; 1 \le j < k \le m \\ h(Y_0, Z_0, t_0) &= X_0, \end{aligned} \tag{9''}$$

whose vector fields denoted by $\widetilde{a}_j$ and $\widetilde{a}_{jk}$, respectively, commute in view of the algebraic assumption on the primary vector fields $a_j$. As before we take any $C^1$–solution $h$ with

$$\det\left(\frac{\partial h}{\partial y}(y, \widetilde{z}, t_0)\right) \neq 0$$

and $\frac{\partial h}{\partial t}(y, \widetilde{z}, t)$ being Lipschitz continuous in $y$ in a neighborhood of $\big(Y_0, (Z_0, 0), t_0\big)$ and seek the solution of (7) in the form

$$X(t) = h\big(Y(t), \widetilde{Z}(t), t\big) \tag{10''}$$

with $Z^{jk}(t) := \int_{t_0}^{t} Z_j \, dZ_k$, $1 \le j < k \le m$.

The random process $Y(t)$ is determined by the ODE

$$\begin{aligned}\dot{Y}(t) = &\left(\frac{\partial h}{\partial y}\big(Y(t),\widetilde{Z}(t),t\big)\right)^{-1}\Bigg[b\Big(h\big(Y(t),\widetilde{Z}(t),t\big),t\Big) - \frac{\partial h}{\partial t}\big(Y(t),\widetilde{Z}(t),t\big)\\
&- \frac{1}{2}\sum_{j,k=1}^{m} q_{jk}(t)\left(\frac{\partial \widetilde{a}_j}{\partial x}\Big(h\big(Y(t),\widetilde{Z}(t),t\big),t\Big)\right)\widetilde{a}_k\Big(h\big(Y(t),\widetilde{Z}(t),t\big),t\Big)\\
&- \frac{1}{2}\sum_{i=1}^{m}\sum_{1\le j<k\le m} Z^j(t)\,q_{ik}(t)\left(\frac{\partial a_i}{\partial x}\Big(h\big(Y(t),\widetilde{Z}(t),t\big),t\Big)\right)\\
&\qquad\qquad \widetilde{a}_{jk}\Big(h\big(Y(t),\widetilde{Z}(t),t\big),t\Big)\\
&- \frac{1}{2}\sum_{1\le i<j\le m}\sum_{k=1}^{m} Z^i(t)\,q_{jk}(t)\left(\frac{\partial \widetilde{a}_{ij}}{\partial x}\Big(h\big(Y(t),\widetilde{Z}(t),t\big),t\Big)\right)\\
&\qquad\qquad \widetilde{a}_{k}\Big(h\big(Y(t),\widetilde{Z}(t),t\big),t\Big)\\
&- \frac{1}{2}\sum_{1\le i<j\le m}\sum_{1\le k<l\le m} Z^i(t)\,Z^k(t)\,q_{jl}(t)\left(\frac{\partial \widetilde{a}_{ik}}{\partial x}\Big(h\big(Y(t),\widetilde{Z}(t),t\big),t\Big)\right)\\
&\qquad\qquad \widetilde{a}_{kl}\Big(h\big(Y(t),\widetilde{Z}(t),t\big),t\Big)\Bigg]\end{aligned} \tag{11''}$$

$$Y(t_0) = Y_0\,.$$

In order to make this precise we assume the following

(C3) The stochastic integrals $Z^{ij}(t) = \int_{t_0}^{t} Z^i\,dZ^j$ are determined and possess the covariation properties
$\big[Z^i, Z^{jk}\big] = \int_0^{(\cdot)} Z^j(s)\,q_{ik}(s)\,ds$
$\big[Z^{ij}, Z^{kl}\big] = \int_0^{(\cdot)} Z^i(s)\,Z^k(s)\,q_{jl}(s)\,ds.$
Moreover, the processes $Z^{ij}$ satisfy the Itô formula for the random functions $h$ considered in ($9''$), i.e., the differentials $dZ^{ij}$ may be replaced by $Z^i\,dZ^j$.

*Remark.* Clearly, the general problem consists in checking the last condition in (C3). Note that by Theorem 3.2 it is fulfilled for all vector fields (independently of the structure of the Lie algebra) if the processes $Z^j$ are elements of $W^{\beta}_{2,\infty}$ for some $\beta > 1/2$.

The rest is completely analogous to the proof of Theorem 7.2.1: Differentiating ($10''$) according to the Itô rule and using the expressions in ($9''$) and ($11''$) for the derivatives one obtains that $X(t)$ is indeed a local solution of (7). The proof of uniqueness in the sense of our solution concept by means of the inverse mapping theorem is similar to the case $m = 1$.

Finally, the case $p > 2$ is a straightforward extension, though the algebraic part is much harder: Here the iterated integrals of the processes $Z^1, \ldots, Z^m$ up to order $p$ are involved. The problem again consists in blowing up the vector fields $a_1, \ldots, a_m$ in order to get an integrable system of PDE for determining $h$ such that the use of the iterated integrals as additional arguments leads through the Itô formula for $h$ to the coefficients $a_j$ at $dZ^j$. An algebraic procedure for determining such a differential system is fully described in Yamato [22]. The remainig part is before. Again, this approach does only work if the iterated integrals satisfy the Itô formula for the random functions under consideration.

## References


[1] Alòs, E., and D. Nualart: An extension of Itô's formula for anticipating processes. *J. Theor. Probab.* 11 (1998), 493-514.

[2] J. Asch and J. Potthoff: Itô's lemma without non–anticipatory conditions. *Probab. Th. Rel. Fields* 88 (1991), 17-46.

[3] Berger, M.A., and V.J. Mizel: An extension of the stochastic integral. *Ann. Probab.* 10 (1982), 435-450.

[4] Doss, H.: Liens entre équations différentielles stochastiques et ordinaires. *Ann. Inst. Henri Poincaré, Probab. et Stat.* 13 (1977), 99-125.

[5] Feyel, D., and A. de La Pradelle: Fractional integrals and Brownian processes. *Preprint.*

[6] Föllmer, H., Calcul d'Itô sans probabilité. *Séminaire de probabilités XV, Lecture Notes in Math.* 850 (1979/80), 143-150.

[7] Ikeda, N., and S. Watanabe: Stochastic differential equations and diffusion processes. *North-Holland Publ. Comp.*, 1981.

[8] Klingenhöfer, F.: Differential equations with fractal noise. *Ph.D. Thesis, University of Jena, Mathematical Institute*, 1999.

[9] Klingenhöfer, F., and M. Zähle: Ordinary differential equations with fractal noise. *Proc. Amer. Math. Soc.* 127 (1999), 1021-1028.

[10] Kuo, H.H., and A. Russek: White noise approach to stochastic integration. *J. Multivariate Analysis* 24 (1988), 218-236.

[11] Lin, S.J.: Stochastic analysis of fractional Brownian motion. *Stochastics and Stochastics Reports* 55 (1995), 121-140.

[12] Lyons, T.: Differential equations driven by rough signals (I): An extension of an equality of L.C. Young. *Math. Research Letters* 1 (1994), 451-464.

[13] Mazja, W., and J. Nagel: Über äquivalente Normierung der anisotropen Funktionräume $H^\mu(\mathbb{R}^n)$. *Beitr. Analysis* 12/5 (1978), 7-18.

[14] Nualart, D.: The Malliavin calculus and related topics. *Springer* 1995.

[15] Pardoux, E.: Applications of anticipating stochastic calculus to stochastic differential equations. *in: Stochastic Analysis and Related Topics II, H. Korezlioglu and A.S. Üstünel Eds., Lecture Notes in Math.* 1444 (1988), 63-105.

[16] Russo, F., and P. Vallois: Forward, backward and symmetric stochastic integration. *Probab. Th. Relat. Fields* 97 (1993), 403-421.

[17] Russo, F., and P. Vallois: The generalized covariation process and Itô formula. *Stoch. Processes Appl.* 59 (1995), 81-104.

[18] Samko, S.G., Kilbas, A.A., and O.I. Marichev: Fractional integrals and derivatives. Theory and applications. *Gordon and Breach*, 1993.

[19] Sussman, H.: On the gap between deterministic and stochastic ordinary differential equations. *Ann. Probab.* 6 (1978), 19-41.

[20] Zähle, M.: Integration with respect to fractal functions and stochastic calculus I. *Probab. Th. Relat. Fields* 111 (1998), 333-374.

[21] Zähle, M.: On the link between fractional and stochastic calculus. In: Stochastic Dynamics, (Eds. H. Crauel and M. Gundlach), *Springer* 1999.

[22] Yamato, Y.: Stochastic differential equations and nilpotent Lie algebras. *Z. Wahr-scheinlichkeitstheorie verw. Geb.* 47 (1979), 213-229.

[23] Young, L.C: An inequality of Hölder type, connected with Stieltjes integration. *Acta Math.* 67 (1936), 251-282.