POTENTIAL SPACES ON FRACTALS

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Abstract. We introduce potential spaces on fractal metric spaces, investigate their embedding theorems, and derive various Besov spaces. Our starting point is that there exists a local, stochastically complete heat kernel satisfying a two-sided estimate on the fractal considered. The results of this paper are among the marvelous consequences of the heat kernel on the fractal.

1. Introduction

The classical Besov spaces $B^s_{pq}(\mathbb{R}^n)$ with $s > 0$ and $1 \le p, q \le \infty$ are closely related to the Gauss-Weierstrass and the Cauchy-Poisson semigroups. This goes back to Taibleson [24] and Flett [9]. Similarly, the fractional Sobolev spaces $H^s_p(\mathbb{R}^n), s \in \mathbb{R}, 1 \le p \le \infty$ may be expressed in terms of these semigroups. In particular, for $s > 0$ they can be interpreted as the potential spaces (see for example [21, Chapter V]). The corresponding Gauss-Weierstrass and Cauchy-Poisson heat kernels are explicitly given on $\mathbb{R}^n$.

For several fractal sets, local Dirichlet forms and sub-Gaussian heat kernel estimates of the corresponding semigroups have been obtained. A new and interesting phenomenon has been discovered that the *walk dimensions* of the heat kernels on these fractal sets are strictly greater than 2. The importance of the walk dimension w is that the number $\frac{w}{2}$ measures the smoothness degree of functions defined on the underlying space. As to the sub-Gaussian estimates of the heat kernels, the reader may refer to the pioneering paper [4] for the Sierpinski gaskets, [3] for the Sierpinski carpets, [10] for nested fractals, and to [13, 18] for a certain class of post-critically finite fractals. The associated Markov processes are Brownian motions on the fractal sets.

In the present paper the striking point is a complete metric space (X, ρ) admitting a heat kernel with respect to a Radon measure μ supported on X. Our main assumption is that the heat kernel is *local* and satisfies two-sided decay estimates (see $(G6)$ and $(G7)$ below). By means of the associated semigroup $\{G_t\}_{t\ge 0}$ we introduce function spaces on X which agree with those mentioned above for $X = \mathbb{R}^n$. As in the Euclidean case we obtain some characterizations and embedding theorems. (This may be applied for nonlinear PDE's on X.) The opposite way is to introduce function spaces on fractals in a more direct way (see for example [17, 25, 26, 22]) and to find heat kernel estimates related to associated p-energy forms. First steps in this direction are [7] for some non-local structure and [28] for the local version on d-sets, when $p = 2$ and the walk dimension is equal to 2.

The paper is organized as follows: in Section 2 we first introduce potential operators of a general *strongly continuous contractive semigroup* $\{G_t\}_{t\ge 0}$ in the same way as for the

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Gauss-Weierstrass semigroup. They coincide with the positive fractional powers of the operators $(I - A_p)^{-1}$ and $(-A_p)^{-1}$ (provided they exist), respectively, where A_p denotes the p-generator of $\{G_t\}_{t\geq 0}$. Then we obtain a formula for the positive fractional powers of $(I - A_p)$ as inverse operators. (The proof is similar to the case of Gauss-Weierstrass semigroups [19].) If $\{G_t\}_{t\geq 0}$ has a local heat kernel satisfying some two-sided decay estimates, the generator A_p may be interpreted as a *p-Laplacian*. Under an integrability condition for the upper bound, the reference measure μ is shown to be a d-measure. Moreover, the potential operators of order α in this case possess integral kernels (locally) equivalent to $\rho(x,y)^{d-\alpha w/2}$, which corresponds to the classical case if $d = n$ and $w = 2$. Thus we may interpret these operators as the *p-Bessel* and *p-Riesz* potential operators.

In Section 3 fractional Sobolev spaces on X are introduced as the Bessel potential spaces for local heat kernels satisfying the decay condition. For the special case $p = 2$ and $0 < \alpha \leq 1$, we obtain characterizations which lead, for d-subsets in $\mathbb{R}^n$, to the well-known Besov spaces introduced in [17] with the method of traces for $w = 2$, $0 < \alpha < 1$, and to the Lipschitz spaces initiated in [16] for $w \geq 2$, $\alpha = 1$.

In Section 4 embeddings of Sobolev spaces in $L_q(\mu)$ spaces or in Hölder spaces are discussed. The classical cases are again included.

Finally, Section 5 complements these contributions by the definition of Besov spaces $B^\alpha_{pq}(\mu)$ related to the semigroup $\{G_t\}_{t\geq 0}$. For $p = q = 2$, they coincide with the corresponding Sobolev spaces, which generalizes the Euclidean case.

2. Bessel and Riesz potentials

2.1. **Assumptions.**

Let (X, ρ) be a separable, complete metric space that is generally interpreted as a bounded or unbounded fractal. Let μ be a locally finite Borel measure with $\operatorname{supp} \mu = X$. Assume that (X, ρ) is *connected* in the sense that (X, ρ) satisfies the *chain condition*, that is there exists a constant $C > 0$ such that, for any two points $x, y \in X$ and for any positive integer n, there is a sequence $\{x_i\}_{i=0}^n$ of points in X with $x_0 = x, x_n = y$, and

$$\rho(x_i, x_{i+1}) \leq C\, n^{-1} \rho(x, y), \quad 0 \leq i \leq n - 1.$$

Let $G(t, x, y)$ be a stochastically complete *heat kernel* or *transition density* on (X, ρ, μ), that is, $G(t, x, y)$ is a real-valued function on $(0, \infty) \times X \times X$ satisfying the following conditions: for all $0 < t < \infty$ and $x, y \in X$,

(G1)(non-negativity): $G(t, x, y) \geq 0$;

(G2)(symmetry): $G(t, x, y) = G(t, y, x)$;

(G3)(semigroup property): $G(s+t, x, y) = \int_X G(s, x, z) G(t, z, y) d\mu(z) \quad (s > 0)$;

(G4)(identity approximation): $\lim\limits_{t\to 0+} \|G_t f - f\|_p = 0$ for any $f \in L_p(\mu)$ where $1 \leq p < \infty$, and $\lim\limits_{t\to 0+} G_t f(x) = f(x)$ for μ-almost all $x \in X$ if $f \in L_\infty(\mu)$;

(G5) (stochastic completeness): $\int_X G(t,x,y))d\mu(y) = 1$.

Here $L_p(\mu) := L_p(X,\rho,\mu)$ is the usual real-valued p-integrable functions space with norm

$$\|f\|_p := \Big(\int_X |f(x)|^p d\mu(x)\Big)^{1/p} \quad (1 \le p < \infty), \quad \text{and } \|f\|_\infty := \operatorname*{ess\,sup}_{x\in X} |f(x)|.$$

The $\{G_t\}_{t\ge 0}$ is the *semigroup* associated with $G(t,x,y)$:

$$G_t f(x) = \int_X G(t,x,y)f(y)d\mu(y) \quad (t>0, x\in X). \tag{2.1}$$

(As usual we set $G_0 = I$, the *identity operator* on $L_p(\mu)$).

Remark: Condition $(G5)$ is important in this paper, and it can not be replaced by a weaker version

$$\int_X G(t,x,y)d\mu(y) \le 1 \quad (t>0, x\in X).$$

Example 2.1 (**Classical case).**

Let $X = \mathbb{R}^n$ with ρ the Euclidean metric and μ be the Lebesgue measure. It is easy to see that the *Gauss-Weierstrass* heat kernel

$$G_{\mathbb{R}^n}(t,x,y) = \Big(\frac{1}{4\pi t}\Big)^{\frac{n}{2}} \exp\Big(-\frac{|x-y|^2}{4t}\Big) \tag{2.2}$$

satisfies $(G1)$-$(G5)$. The *Cauchy-Poisson* heat kernel

$$P_{\mathbb{R}^n}(t,x,y) = C_n t^{-n}(1+t^{-2}|x-y|^2)^{-\frac{n+1}{2}} \qquad \Big(C_n = \pi^{-\frac{n+1}{2}}\Gamma((n+1)/2)\Big)$$

also satisfies $(G1)$-$(G5)$.

Example 2.2 (**Fractal case).**

1 (Brownian motion on the Sierpinski carpet). Let X be a Sierpinski carpet in $\mathbb{R}^n$ with the Euclidean metric, and let μ be the d-dimensional Hausdorff measure on X. Barlow and Bass [3] showed that there exists a heat kernel on X satisfying

$$a_1\, t^{-\frac{d}{w}} \exp\Big(-b_1\,(t^{-1/w}|x-y|)^\gamma\Big) \le G(t,x,y) \le a_2\, t^{-\frac{d}{w}} \exp\Big(-b_2\,(t^{-1/w}|x-y|)^\gamma\Big) \tag{2.3}$$

for all $x, y \in X$ and all $0 < t < \operatorname{diam}(X)$, where $w > 2$ and $\gamma = \frac{w}{w-1}$, $a_i, b_i > 0 (i = 1,2)$. The *Barlow-Bass* heat kernel satisfies $(G1)$-$(G5)$

2 (Stable-like processes). For each $0 < \sigma < 1$, there is a function $p^{(\sigma)}(t,x,y)$ on the Sierpinski carpet in $\mathbb{R}^n$ satisfying $(G1)$-$(G5)$, and

$$a_1\, t^{-\frac{d}{\sigma w}} \Big(1+t^{-\frac{1}{\sigma w}}|x-y|\Big)^{-(d+\sigma w)} \le p^{(\sigma)}(t,x,y) \le a_2\, t^{-\frac{d}{\sigma w}} \Big(1+t^{-\frac{1}{\sigma w}}|x-y|\Big)^{-(d+\sigma w)} \tag{2.4}$$

for all $x, y \in X$ and all $0 < t < \operatorname{diam}(X)$, where $a_1, a_2 > 0$, see for example [5] or [15].

Notation. Throughout this paper we denote by C a general constant and let $r_0 = \operatorname{diam}(X) \in (0,\infty]$. Two non-negative function f and g are *equivalent*, denoted by $f \cong g$, if $C^{-1}f(x) \le g(x) \le Cf(x)$ for all $x \in X$ and some $C > 0$. We mean by the "classical case" $X = \mathbb{R}^n$, $\rho(x,y) = |x-y|$ and μ is the Lebesgue measure on $\mathbb{R}^n$, see Example 2.1.

By $(G4)$, $\{G_t\}_{t\geq 0}$ is *strongly continuous* on $L_p(\mu)$ $(1 \leq p < \infty)$:

$$\lim_{t\to t_0} \|G_t f - G_{t_0} f\|_p = 0 \quad (f \in L_p(\mu), t_0 \geq 0),$$

and by $(G5)$, $\{G_t\}_{t\geq 0}$ is *contractive*:

$$\|G_t f\|_p \leq \|f\|_p \quad (t \geq 0, f \in L_p(\mu), 1 \leq p \leq \infty).$$

Therefore, there exists an *infinitesimal generator* A_p of $\{G_t\}_{t\geq 0}$ on $L_p(\mu)(1 \leq p < \infty)$:

$$A_p f = \lim_{t\to 0+} \frac{G_t f - f}{t}, \quad \text{strongly in } L_p(\mu) \tag{2.5}$$

with *domain* $\mathcal{D}(A_p)$, the space of all functions f such that the limit in (2.5) exists. Note that $\mathcal{D}(A_p)$ is dense in $L_p(\mu)$, see for example [27, p.237].

In the following we will repeatedly specify to $L_2(\mu)$. For simplicity we denote $A := A_2$ and $\mathcal{D}(A) := \mathcal{D}(A_2)$. By $(G2)$, we see that A is *self-adjoint*:

$$(Af, g) = (f, Ag) \quad (f, g \in \mathcal{D}(A)),$$

where $(\ ,\ )$ is the inner product in $L_2(\mu)$

$$(f, g) := \int_X f(x)g(x)\, d\mu(x).$$

Moreover, the linear operator A is *non-positive definite*:

$$(Af, f) \leq 0 \quad (f \in \mathcal{D}(A)) \tag{2.6}$$

since, by $(G5)$ and $(G1)$,

$$(Af, f) = -\lim_{t\to 0} \mathcal{E}_t(f, f) \leq 0 \quad (f \in \mathcal{D}(A)), \tag{2.7}$$

where

$$\mathcal{E}_t(f, g) := \frac{1}{2t} \int_X \int_X (f(y) - f(x))(g(y) - g(x))G(t, x, y)d\mu(y)d\mu(x). \tag{2.8}$$

Thus $-A$ admits a unique spectral resolution:

$$-Af = \int_0^\infty \lambda\, dE_\lambda f \quad (f \in \mathcal{D}(A)), \tag{2.9}$$

see for example [27, p.313]. Note that

$$G_t f = \int_0^\infty e^{-\lambda t} dE_\lambda f \quad (f \in L_2(\mu),\ t \geq 0). \tag{2.10}$$

For $\alpha \in \mathbb{R}$, we define

$$\begin{aligned} \mathcal{D}((-A)^\alpha) &= \left\{ f \in L_2(\mu) : \int_0^\infty \lambda^{2\alpha} d(E_\lambda f, f) < \infty \right\}, \\ (-A)^\alpha f &= \int_0^\infty \lambda^\alpha dE_\lambda f \quad (f \in \mathcal{D}((-A)^\alpha)). \end{aligned} \tag{2.11}$$

The unique Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with A is determined by

$$\begin{aligned} \mathcal{F} &= \mathcal{D}\left((-A)^{1/2}\right), \\ \mathcal{E}(f, g) &= \left((-A)^{1/2} f, (-A)^{1/2} g\right) \quad (f, g \in \mathcal{F}), \end{aligned} \tag{2.12}$$

see for example [11]. We may characterize $(\mathcal{E},\mathcal{F})$ in terms of $G(t,x,y)$ as follows

$$\begin{aligned} \mathcal{F} &= \Big\{ f \in L_2(\mu) : \lim_{t\to 0} \mathcal{E}_t(f,f) < \infty \Big\}, \\ \mathcal{E}(f,g) &= \lim_{t\to 0} \mathcal{E}_t(f,g) \quad (f,g \in \mathcal{F}). \end{aligned} \tag{2.13}$$

Note that $(G5)$ plays an important part in obtaining (2.13).

2.2. **Potentials associated with semigroups.**

In the sequel we will study two distinct kinds of potentials associated with semigroups as above. We first consider the general case of a strongly continuous contractive semigroup $\{G_t\}_{t\geq 0}$ on the Banach space $L_p(\mu),\ p \geq 1$ (where the kernel assumption in Subsection 2.1 is not needed).

Definition 2.3. For $\alpha > 0$ and $1 \leq p \leq \infty$, the *potential operators* of order α of $f \in L_p(\mu)$ are defined by

$$J_\mu^\alpha f(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} G_t f(x) dt, \tag{2.14}$$

$$I_\mu^\alpha f(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} G_t f(x) dt \quad (x \in X). \tag{2.15}$$

Remarks. 1. Note that $I_\mu^\alpha f$ may not be well-defined for some α and f, if X is bounded. For example, if $f = 1$ then $I_\mu^\alpha f = \infty$ for any $\alpha > 0$. However, $J_\mu^\alpha f$ is well-defined for any $\alpha > 0$ and any $f \in L_p(\mu), 1 \leq p \leq \infty$, because $J_\mu^\alpha f \in L_p(\mu)$ for $f \in L_p(\mu)$ due to the fact that $\|J_\mu^\alpha f\|_p \leq \|f\|_p$. For this reason we will mostly work on potential operators J_μ^α.

2. For the classical case, if $G(t,x,y)$ is the Gauss-Weierstrass heat kernel, then J_μ^α and I_μ^α defined as in Definition 2.3 are the Bessel and Riesz potential operators respectively, see for example [21] or [19, 20]. For the Sierpinski carpet X, in order that J_μ^α and I_μ^α be Bessel and Riesz potential operators, we take for $G(t,x,y)$ the Barlow-Bass heat kernel, see Subsection 2.3 below.

The operator J_μ^α or I_μ^α in Definition 2.3 may be interpreted as the *fractional power of order* $\frac{\alpha}{2}$ of the resolvent $(I - A_p)^{-1}$ or of $(-A_p)^{-1}$ (should it exist) by the following arguments. We first consider the case $p = 2$ with symmetric operators G_t, where such powers are introduced by means of the spectral resolution.

Proposition 2.4. *Let J_μ^α be defined as in (2.14), and let A be the generator of a symmetric strongly continuous contractive semigroup $\{G_t\}_{t\geq 0}$ on $L_2(\mu)$. Then the following holds on $L_2(\mu)$*

$$J_\mu^\alpha = (I - A)^{-\alpha/2} \quad (\alpha > 0). \tag{2.16}$$

Proof. Let $\alpha > 0$. By (2.10) and Fubini's theorem, we have that

$$\begin{aligned}\frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}G_t f dt &= \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}\Big(\int_0^\infty e^{-\lambda t}dE_\lambda f\Big)dt \\ &= \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty \Big(\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-(1+\lambda)t}dt\Big)dE_\lambda f \\ &= \int_0^\infty (1+\lambda)^{-\frac{\alpha}{2}}dE_\lambda f = (I-A)^{-\alpha/2}f, \quad f\in L_2(\mu),\end{aligned}$$

which combines with (2.14) to yield (2.16). □

For general $1 \le p \le \infty$, we may use the notation

$$(I-A_p)^{-\alpha/2} := J_\mu^\alpha\ ,$$

since $(I-A_p)^{-1} = J_\mu^2$, and

$$J_\mu^{\alpha_1+\alpha_2} = J_\mu^{\alpha_1}\circ J_\mu^{\alpha_2} \quad (\alpha_1,\alpha_2>0).$$

The last equality follows from (2.14) and the semigroup property of $\{G_t\}_{t\ge 0}$. One can also use the notation

$$(-A_p)^{-\alpha/2} := I_\mu^\alpha$$

by similar arguments for suitable α and f.
(Note that (2.16) may formally be obtained as follows. Since

$$(1+u)^{-\frac{\alpha}{2}} = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-(1+u)t}dt \quad (\alpha>0), \tag{2.17}$$

we replace u by $-A$ and then use $G_t = e^{At}$, to obtain that

$$\begin{aligned}(I-A)^{-\frac{\alpha}{2}} &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}\, e^{At}dt \\ &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}\, G_t dt = J_\mu^\alpha.\end{aligned}$$

Similarly, we replace $1+u$ by u in (2.17), and then let $u=-A$, to obtain that

$$\begin{aligned}(-A)^{-\frac{\alpha}{2}} &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{At}dt \\ &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\frac{\alpha}{2}-1}G_t dt = I_\mu^\alpha.\ )\end{aligned}$$

We will see that there exists an explicit formula for the *inversion* of J_μ^α for any $\alpha>0$ and $1\le p\le\infty$, as in the classical case (cf. [19, 20]). For this, define

$$D_\varepsilon^\alpha = \frac{1}{\chi(\alpha/2,l)}\int_\varepsilon^\infty t^{-\frac{\alpha}{2}-1}(I-e^{-t}\, G_t)^l dt \quad (\varepsilon>0) \tag{2.18}$$

for $\alpha>0$, where $l=[\alpha/2]+1$ ($[\alpha/2]$ is the integer part of $\alpha/2$) and

$$\chi(\alpha/2,l) = \int_0^\infty s^{-\frac{\alpha}{2}-1}(1-e^{-s})^l ds < \infty \quad (0<\alpha/2<l). \tag{2.19}$$

Theorem 2.5. *Let $\{G_t\}_{t\geq 0}$ be a strongly continuous contractive semigroup on $L_p(\mu)$ and $\alpha > 0$. If $1 \leq p < \infty$, the left inverse of J_μ^α exists in the following sense:*

$$\lim_{\varepsilon\to 0} \|D_\varepsilon^\alpha J_\mu^\alpha f - f\|_p = 0 \tag{2.20}$$

for any $f \in L_p(\mu)$. Moreover, if $p = \infty$ then

$$\lim_{\varepsilon\to 0} D_\varepsilon^\alpha J_\mu^\alpha f(x) = f(x) \tag{2.21}$$

for μ-almost all $x \in X$ and $f \in L_\infty(\mu)$.

Proof. The proof is similar to the classical case, see for example [19, Theorem 20.4, p. 260-261] or [1, Theorem 1]. For the reader's convenience, we sketch the proof. Let $f \in L_p(\mu)$. By (2.18), we see that for $f \in L_p(\mu)$ $(1 \leq p \leq \infty)$,

$$D_\varepsilon^\alpha J_\mu^\alpha f = \frac{1}{\chi(\alpha/2, l)} \int_\varepsilon^\infty t^{-\frac{\alpha}{2}-1} \Big\{ \sum_{k=0}^{l} (-1)^k \binom{l}{k} e^{-kt} G_{kt}(J_\mu^\alpha f) \Big\} dt. \tag{2.22}$$

By (2.14) and the semigroup property,

$$\begin{aligned} e^{-kt} G_{kt}(J_\mu^\alpha f) &= \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty s^{\frac{\alpha}{2}-1} e^{-(s+kt)} G_{s+kt} f \, ds \\ &= \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty ((s-kt)_+)^{\frac{\alpha}{2}-1} e^{-s} G_s f \, ds, \end{aligned}$$

where

$$a_+ = \begin{cases} a, & a \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, it follows from (2.22) that

$$D_\varepsilon^\alpha J_\mu^\alpha f = \frac{1}{\chi(\alpha/2, l)\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-s} G_s f \Big\{ \sum_{k=0}^{l} (-1)^k \binom{l}{k} \psi_{k,\varepsilon}(s) \Big\} ds, \tag{2.23}$$

where

$$\psi_{k,\varepsilon}(s) := \int_\varepsilon^\infty t^{-\frac{\alpha}{2}-1} ((s-kt)_+)^{\frac{\alpha}{2}-1} dt.$$

It is not hard to calculate that

$$\psi_{k,\varepsilon}(\varepsilon s) = \frac{2}{\alpha \varepsilon s} ((s-k)_+)^{\frac{\alpha}{2}} \quad (s, k \geq 0).$$

Therefore, we have from (2.23) that

$$D_\varepsilon^\alpha J_\mu^\alpha f = \int_0^\infty K_{\frac{\alpha}{2},l}(s) \, e^{-\varepsilon s} G_{\varepsilon s} f \, ds, \tag{2.24}$$

where

$$K_{\beta,l}(s) = \frac{1}{\chi(\beta, l)\Gamma(\beta+1)} \sum_{k=0}^{l} (-1)^k \binom{l}{k} s^{-1} ((s-k)_+)^\beta$$

for any $\beta > 0$ with $l = [\beta] + 1$. The function $K_{\beta,l}$ has the following properties: $K_{\beta,l} \in L_1(0,\infty)$, and

$$\int_0^\infty K_{\beta,l}(s) ds = 1,$$

see for example [19, Lemma 10.47, p. 158]. Hence, we see from (2.24) that, using the strong continuity and the dominated convergence theorem,

$$\|D_\varepsilon^\alpha J_\mu^\alpha f - f\|_p \le \int_0^\infty |K_{\frac{\alpha}{2},l}(s)| \cdot \|e^{-\varepsilon s} G_{\varepsilon s} f - f\|_p \, ds \to 0$$

as $\varepsilon \to 0$, for any $f \in L_p(\mu)$ and $1 \le p < \infty$.
Now let $p = \infty$ and $f \in L_\infty(\mu)$. Since $\|e^{-\varepsilon s} G_{\varepsilon s} f\|_\infty \le \|f\|_\infty$ for $s \ge 0$, and by the continuity,

$$\lim_{\varepsilon \to 0} e^{-\varepsilon s} G_{\varepsilon s} f(x) = f(x) \ (s > 0)$$

for μ-almost all $x \in X$, we see from (2.24) that $\lim_{\varepsilon \to 0} D_\varepsilon^\alpha J_\mu^\alpha f(x) = f(x)$ for μ-almost all $x \in X$ by using again the dominated convergence theorem, which proves (2.21). □

Remarks. 1. Note that for $p = 2$, Theorem 2.5 can simply be obtained by using the spectral resolution, since

$$(J_\mu^\alpha)^{-1} = (I - A)^{\alpha/2} = \frac{1}{\chi(\alpha/2, l)} \int_0^\infty t^{-\frac{\alpha}{2}-1} (I - e^{-t} G_t)^l dt. \tag{2.25}$$

The Riesz potential operator I_μ^α also has the inversion for $\alpha > 0$ and $1 \le p < \infty$, see for example [20, (5.85), p.121]; in particular, for $p = 2$ we have that

$$\left(I_\mu^\alpha\right)^{-1} = (-A)^{\alpha/2} = \frac{1}{\chi(\alpha/2, l)} \int_0^\infty t^{-\frac{\alpha}{2}-1} (I - G_t)^l dt, \ (l = [\alpha/2] + 1). \tag{2.26}$$

2. For $0 < \alpha < 2$, the formulas (2.25) and (2.26) are called the *Balakrishan formulas*, see for example [27, p.260]. Note that (2.25) or (2.26) can also be formally obtained by using the fact that

$$(1 + u)^{\alpha/2} = \frac{1}{\chi(\alpha/2, l)} \int_0^\infty t^{-\frac{\alpha}{2}-1} (1 - e^{-t(1+u)})^l dt, \ 0 < \alpha/2 < l,$$

and replacing u by $-A$ or $1 + u$ by $-A$.

For $\alpha > 0$, let D_ε^α be defined as in (2.18) with $l = [\alpha/2] + 1$. Define the linear operator D^α by

$$\lim_{\varepsilon \to 0} \|D_\varepsilon^\alpha f - D^\alpha f\|_p = 0 \tag{2.27}$$

for suitable $f \in L_p(\mu)$ where $1 \le p < \infty$. Then D^α can be interpreted as $(I - A_p)^{\alpha/2}$.

2.3. **Local heat kernels with decay conditions.**

We now turn back to the assumptions (G1-G5). In order to study the Bessel potentials J_μ^α in more detail, we need more conditions on the heat kernel $G(t, x, y)$. We say that a heat kernel $G(t, x, y)$ is *local*, and satisfies a two-sided *estimate* respectively, if

(G6)(locality): : $\lim_{t \to 0} t^{-1} \, G(t, x, y) = 0$ for any $x, y \in X$ $(x \ne y)$,

(G7)(estimate): $t^{-\frac{d}{w}} \, \Phi_1 \left(t^{-\frac{1}{w}} \rho(x, y)\right) \le G(t, x, y) \le t^{-\frac{d}{w}} \, \Phi_2 \left(t^{-\frac{1}{w}} \rho(x, y)\right)$
for all $x, y \in X$, $0 < t < r_0$ and some $d > 0, w \ge 2$, where $r_0 = \mathrm{diam}(X)$ ($r_0 = \infty$ if X is unbounded), and $\Phi_i (i = 1, 2)$ are bounded, decreasing functions on $[0, \infty)$.

For X *bounded*, as a complementary condition to ($G7$), we assume that

(G8)(large-time behavior): $G(t,x,y) \leq C\ t^{-\frac{d}{w}} \exp(t/2)$ for all $x, y \in X$ and all $t > r_0$, where $C > 0$.

Condition ($G8$) is very weak and can be obtained from the Nash inequality, see [6, Theorem (2.1), p. 251] (by taking $\delta = 1/2$) if X is bounded. Under these conditions the operator A_p is local and will be interpreted as *p-Laplacian.*

Clearly the Gauss-Weierstrass heat kernel $G_{\mathbb{R}^n}(t,x,y)$ in (2.2) satisfies ($G6$), and ($G7$) with $d = n, w = 2$,

$$\Phi_i(s) = (4\pi)^{-n/2} \exp(-s^2/4) \quad (i = 1, 2),$$

whilst the Cauchy-Poisson heat kernel satisfies ($G7$) with $d = n, w = 1$,

$$\Phi_i(s) = C_n(1+s^2)^{-\frac{n+1}{2}} \quad (s > 0) \quad (i = 1, 2),$$

but it is not local.

The Barlow-Bass heat kernel on the Sierpinski carpet satisfies ($G7$) with

$$\Phi_i(s) = a_i \exp(-b_i\ s^{\gamma}), \quad i = 1, 2. \tag{2.28}$$

It is easy to see from (2.28) that ($G6$) holds, since for $x, y \in X (x \neq y)$,

$$\begin{aligned} 0 &\leq t^{-1} G(t,x,y) \leq t^{-1-\frac{d}{w}} \Phi_2(t^{-\frac{1}{w}} \rho(x,y)) \\ &= \rho(x,y)^{-(d+w)} s^{d+w} \Phi_2(s) \quad \left(s = t^{-\frac{1}{w}} \rho(x,y)\right) \\ &\to 0 \end{aligned}$$

as $t \to 0+$ (or $s \to \infty$).

The condition ($G6$) may be dropped if the heat kernel $G(t,x,y)$ satisfies ($G7$) with

$$\lim_{s \to \infty} s^{d+w} \Phi_2(s) = 0. \tag{2.29}$$

For a stable-like process on the Sirepinski carpet in $\mathbb{R}^n$, the heat kernel $p^{(\sigma)}(t,x,y)$ satisfies ($G7$) with w replaced by σw, and

$$\Phi_i(s) = a_i\,(1+s)^{-(d+\sigma w)},\ s \geq 0,\ i = 1, 2, \tag{2.30}$$

see Example 2.2. Clearly $p^{(\sigma)}(t,x,y)$ is not local.

Condition ($G7$), together with the following condition

$$\int_0^{\infty} s^{d-1} \Phi_2(s) ds < \infty, \tag{2.31}$$

will imply that μ is a *d-measure*, that is

$$C^{-1} r^d \leq \mu(B(x,r)) \leq C\ r^d \tag{2.32}$$

for all $x \in G$ and all $0 < r < r_0$, where $C > 0$ and $B(x,r) = \{y \in X : \rho(y,x) < r\}$ is the ball in X with center x and radius r. More precisely,

Proposition 2.6. *Let (X,ρ,μ) be a connected metric measure space endowed with a stochastically complete heat kernel $G(t,x,y)$ satisfying $(G7)$ with (2.31). Then μ is a d-measure.*

Proof. This result for $r_0=\infty$ (that is X is unbounded) was obtained in [12, Theorem 3.2, p. 2071]. We only consider $r_0<\infty$. The proof is essentially the same. In fact, similar to [12, (3.3), p.2071], we can obtain that there is a $C>0$ such that

$$\mu(B(x,r))\le C\,r^d \tag{2.33}$$

for all $x\in G$ and $0<r\le c_1$, where $c_1=\min(r_0,r_0^{1/w})$. Without loss of generality we assume $c_1<r_0$. Noting that $\mu(X)<\infty$ since X is bounded, we see that (2.33) also holds for all $c_1<r\le r_0$, by changing the constant C when necessary. Thus (2.33) holds for all $0<r\le r_0$ and all $x\in X$ (in fact, (2.33) also holds for all $r_0<r<\infty$ since μ is supported on X).

In order to show the opposite inequality, we note that there is a small number $\varepsilon_0\in(0,r_0^{1-w})$ such that

$$\int_{X\setminus B(x,r)}G(t,x,y)d\mu(y)\le\frac{1}{2}$$

for all $x\in X$ and all $0<r\le r_0$, if $0<t\le\varepsilon_0 r^w$, by using (2.31) and (2.33)(cf [12, (3.6), p.2072]), and so

$$\int_{B(x,r)}G(t,x,y)d\mu(y)\ge\frac{1}{2}.$$

Therefore

$$\mu(B(x,r))\ge\frac{1}{2}\Big(\sup_{y\in B(x,r)}G(t,x,y)\Big)^{-1} \tag{2.34}$$

for all $x\in X$, $0<r\le r_0$ and all $0<t\le\varepsilon_0 r^w$. In particular, we take $t=\varepsilon_0 r^w$ and then use $(G7)$ to obtain that

$$\sup_{y\in B(x,r)}G(t,x,y)\le\Phi_2(0)t^{-\frac{d}{w}}=C\,r^{-d}$$

for all $x\in X$ and $0<r\le r_0$. Thus, it follows from (2.34) that

$$\mu(B(x,r))\ge C\,r^d$$

for all $x\in X$ and $0<r\le r_0$. □

From now on we assume that

(Main Assumption): *There exists a local, stochastically complete heat kernel $G(t,x,y)$ on (X,ρ,μ), that is, there is a function $G(t,x,y)$ on $(0,\infty)\times X\times X$ satisfying $(G1)$-$(G6)$.*

Condition $(G7)$ with (2.31) guarantees the existence of integral kernels of J^α_μ and I^α_μ, with two-sided estimates.

Proposition 2.7. *Assume that $G(t,x,y)$ additionally satisfies $(G7)$ with the integral condition (2.31), and that $0<\alpha<\frac{2}{w}d$. Then the potential operator J^α_μ has an integral kernel, that is*

$$J^\alpha_\mu f(x)=\int_X B^\alpha_\mu(x,y)f(y)d\mu(y)\quad(x\in X), \tag{2.35}$$

for $f \in L_p(\mu)$ *and* $1 \le p \le \infty$*, where*

$$B_\mu^\alpha(x,y) = \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}G(t,x,y)dt \quad (x,y \in X, x \ne y). \tag{2.36}$$

Moreover, there exists some $C > 0$ *such that*

$$B_\mu^\alpha(x,y) \le C\,\rho(x,y)^{-(d-\alpha w/2)} \tag{2.37}$$

for all $x, y \in X$*, and*

$$B_\mu^\alpha(x,y) \ge C^{-1}\,\rho(x,y)^{-(d-\alpha w/2)} \tag{2.38}$$

for all $x,y \in X$ *with* $0 < \rho(x,y) \le \min(M_0, r_0)$*, where* $M_0 \in (0,\infty)$ *is any fixed number.*

Proof. First let X be bounded. For any $x, y \in X (x \ne y)$, We see from $(G7)$ that

$$\begin{aligned}\int_0^{r_0} t^{\frac{\alpha}{2}-1}e^{-t}G(t,x,y)dt &\le \int_0^{r_0} t^{\frac{\alpha}{2}-1}G(t,x,y)dt\\ &\le \int_0^{r_0} t^{\frac{\alpha}{2}-1-\frac{d}{w}}\Phi_2\left(t^{-\frac{1}{w}}\rho(x,y)\right)dt\\ &= w\,\rho(x,y)^{-(d-\frac{\alpha}{2}w)}\int_{\rho(x,y)r_0^{-1/w}}^\infty s^{d-\frac{\alpha}{2}w-1}\Phi_2(s)ds\\ &\le C\ \rho(x,y)^{-(d-\frac{\alpha}{2}w)}\end{aligned}$$

since, by using the monotonicity of Φ_2,

$$\begin{aligned}\int_{\rho(x,y)r_0^{-1/w}}^\infty s^{d-\frac{\alpha}{2}w-1}\Phi_2(s)ds &\le \int_0^\infty s^{d-\frac{\alpha}{2}w-1}\Phi_2(s)ds\\ &= \int_0^1 s^{d-\frac{\alpha}{2}w-1}\Phi_2(s)ds + \int_1^\infty s^{d-\frac{\alpha}{2}w-1}\Phi_2(s)ds\\ &\le \Phi_2(0)\int_0^1 s^{d-\frac{\alpha}{2}w-1}ds + \int_1^\infty s^{d-1}\Phi_2(s)ds\\ &< \infty.\end{aligned}$$

On the other hand, by $(G8)$,

$$\int_{r_0}^\infty t^{\frac{\alpha}{2}-1}e^{-t}G(t,x,y)dt \le C\int_{r_0}^\infty t^{\frac{\alpha}{2}-1-\frac{d}{w}}e^{-t/2}dt \le C.$$

Therefore,

$$\begin{aligned}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}G(t,x,y)dt &= \int_0^{r_0} t^{\frac{\alpha}{2}-1}e^{-t}G(t,x,y)dt + \int_{r_0}^\infty t^{\frac{\alpha}{2}-1}e^{-t}G(t,x,y)dt\\ &\le C\left(1+\rho(x,y)^{-(d-\frac{\alpha}{2}w)}\right)\\ &\le C\,\rho(x,y)^{-(d-\frac{\alpha}{2}w)}\end{aligned}$$

for all $x, y \in X(x \neq y)$ since X is bounded and $d > \alpha w/2$. Thus (2.37) follows. On the other hand, by $(G7)$ and noting the fact that $\rho(x,y) \leq r_0$, we have that

$$\begin{aligned}
\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}G(t,x,y)dt &\geq e^{-r_0}\int_0^{r_0} t^{\frac{\alpha}{2}-1}G(t,x,y)dt \\
&\geq e^{-r_0}\int_0^{r_0} t^{\frac{\alpha}{2}-1-\frac{d}{w}}\Phi_1\left(t^{-\frac{1}{w}}\rho(x,y)\right)dt \\
&= w\, e^{-r_0}\rho(x,y)^{-(d-\frac{\alpha}{2}w)}\int_{\rho(x,y)r_0{}^{-1/w}}^\infty s^{d-\frac{\alpha}{2}w-1}\Phi_1(s)ds \\
&\geq w\, e^{-r_0}\rho(x,y)^{-(d-\frac{\alpha}{2}w)}\int_{r_0^{1-1/w}}^\infty s^{d-\frac{\alpha}{2}w-1}\Phi_1(s)ds \\
&\geq C^{-1}\rho(x,y)^{-(d-\frac{\alpha}{2}w)}.
\end{aligned}$$

Therefore (2.38) follows if X is bounded. In a similar way, we can obtain (2.37) and (2.38) if X is unbounded. The remaining statement follows from (2.14) and (2.1). □

Assume that $G(t,x,y)$ satisfies $(G7)$ with (2.31). It is not hard to see from (2.36) and $(G1)$-$(G5)$ that, for $x, y \in X(x \neq y)$ and $0 < \alpha < \frac{2}{w}d$,

- $B_\mu^\alpha(x,y) \geq 0$. • $B_\mu^\alpha(x,y) = B_\mu^\alpha(y,x)$.
- $B_\mu^{\alpha_1+\alpha_2}(x,y) = \int_X B_\mu^{\alpha_1}(x,z)B_\mu^{\alpha_2}(z,y)d\mu(z) \quad (\alpha_1, \alpha_2 > 0, \alpha_1+\alpha_2 < \frac{2}{w}d)$.
- $\lim_{\alpha\to 0+} \|J_\mu^\alpha f - f\|_p = 0 \quad (f \in L_p(\mu), 1 \leq p < \infty)$. • $\int_X B_\mu^\alpha(x,y)d\mu(y) = 1$.

Proposition 2.8. *Let X be unbounded. Assume that $G(t,x,y)$ additionally satisfies $(G7)$ with the integral condition (2.31), and that $0 < \alpha < \frac{2}{w}d$. Then the potential operator I_μ^α has an integral kernel, that is*

$$I_\mu^\alpha f(x) = \int_X R_\mu^\alpha(x,y)f(y)d\mu(y), \tag{2.39}$$

whith

$$R_\mu^\alpha(x,y) = \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty t^{\frac{\alpha}{2}-1}G(t,x,y)dt \quad (x,y \in X, x \neq y). \tag{2.40}$$

Moreover, we have that

$$R_\mu^\alpha(x,y) \cong \rho(x,y)^{-(d-\alpha w/2)} \quad (x,y \in X, x \neq y). \tag{2.41}$$

Proof. The proof is similar to that of Proposition 2.7. We omit the details. □

Definition 2.9. Assume that $G(t,x,y)$ satisfies $(G1)$-$(G7)$ on a measure metric space (X,ρ,μ) with the chain condition. For $\alpha > 0$ and $1 \leq p \leq \infty$, the J_μ^α and I_μ^α defined as in (2.14) and (2.15) are termed the *p-Bessel* and *p-Riesz potential operators* respectively.

The operators $D^\alpha = (I - A)^{\alpha/2}$ and $(-A)^{\alpha/2}$ may be interpreted as *Bessel* and *Riesz fractional derivatives*, respectively.

3. Sobolev spaces

Assume that $G(t,x,y)$ satisfies $(G1)$-$(G7)$. For each $\alpha>0$, let J_μ^α be the Bessel potential operator defined as in (2.14). In this section we introduce (fractional) Sobolev-type spaces on (X,ρ,μ). For $0<\alpha\le 1$ and $p=2$, these spaces are shown to coincide with Lipschitz spaces initiated by Jonsson and Wallin [16, 17]. If X is an open subset of $\mathbb{R}^n$ with "nice" boundary, they are equivalent to the usual classical Sobolev spaces. These function spaces arise as the domains of $(I-A_p)^{\alpha/2}$, and play an important rôle in studying nonlinear (fractional) PDE's on (X,ρ,μ). Their embedding theorems will be given in the next section. Note that Theorem 2.5 is crucial to our argument.

For any $\alpha>0$ and $1\le p\le\infty$, we see from (2.20) and (2.21) that the Bessel potential operator $J_\mu^\alpha : L_p(\mu)\to L_p(\mu)$ is one-to-one.

Definition 3.1. Let $\alpha>0$ and $1\le p\le\infty$. The *Sobolev space* (or *Bessel potential space*) $H_p^\alpha(\mu):=H_p^\alpha(X,\rho,\mu,G)$ on (X,ρ,μ) is the image of $L_p(\mu)$ under J_μ^α. The *norm* of $f=J_\mu^\alpha\varphi\in H_p^\alpha(\mu)$ is

$$\|f\|_{H_p^\alpha(\mu)}=\|\varphi\|_p. \tag{3.1}$$

Strichartz [22] introduced the Sobolev-type *spaces* $L_s^p(X)$ for $s>0$ and $1<p<\infty$. It is easy to see that

$$L_s^p(X)=H_p^{2s/w}(\mu)\quad (s>0, 1<p<\infty). \tag{3.2}$$

We may characterize $H_p^\alpha(\mu)$ in an alternative way. Let $\alpha>0$ and $1\le p<\infty$. Define the space $L_{p,p}^\alpha(\mu)$ by

$$L_{p,p}^\alpha(\mu)=\{f\in L_p(\mu): D^\alpha f\in L_p(\mu)\} \tag{3.3}$$

where $D^\alpha f$ is defined as in (2.27). The *norm* of $f\in L_{p,p}^\alpha(\mu)$ is

$$\|f\|_{L_{p,p}^\alpha(\mu)}=\|f\|_p+\|D^\alpha f\|_p\ .$$

Proposition 3.2. *Let $\alpha>0$ and $1\le p<\infty$. Then $H_p^\alpha(\mu)=L_{p,p}^\alpha(\mu)$ with equivalent norms:*

$$\|f\|_{H_p^\alpha(\mu)}\le\|f\|_{L_{p,p}^\alpha(\mu)}\le 2\|f\|_{H_p^\alpha(\mu)}.$$

Proof. Assume that $f\in H_p^\alpha(\mu)$. Write $f=J_\mu^\alpha\varphi$ for some $\varphi\in L_p(\mu)$. It follows from (2.20) and (2.27) that

$$D^\alpha f=\lim_{\varepsilon\to 0}D_\varepsilon^\alpha f=\lim_{\varepsilon\to 0}D_\varepsilon^\alpha J_\mu^\alpha\varphi=\varphi$$

in the L_p-norm since $1\le p<\infty$. Thus,

$$\|f\|_p+\|D^\alpha f\|_p=\|J_\mu^\alpha\varphi\|_p+\|\varphi\|_p\le 2\|\varphi\|_p=2\|f\|_{H_p^\alpha(\mu)},$$

proving that $H_p^\alpha(\mu)$ is embedded in $L_{p,p}^\alpha(\mu)$. Conversely, assume that $f\in L_{p,p}^\alpha(\mu)$. Let $\varphi=D^\alpha f\in L_p(\mu)$. Then $f=J_\mu^\alpha\varphi$; this is because, for any $g\in L^{p'}(\mu)$ (p' is the conjugate of p) we have

$$\begin{aligned}(J_\mu^\alpha\varphi,g)&=(\varphi,J_\mu^\alpha g)=\lim_{\varepsilon\to 0}(D_\varepsilon^\alpha f,J_\mu^\alpha g)\\&=\lim_{\varepsilon\to 0}(f,D_\varepsilon^\alpha J_\mu^\alpha g)=(f,g).\end{aligned}$$

(Note that the last equality still holds if $p = 1$ by using (2.21) and the dominated convergence theorem.) Therefore, we see that $f = J^\alpha_\mu \varphi \in H^\alpha_p(\mu)$, and

$$\|f\|_{H^\alpha_p(\mu)} = \|\varphi\|_p = \|D^\alpha f\|_p \le \|f\|_{L^\alpha_{p,p}(\mu)},$$

showing that $L^\alpha_{p,p}(\mu)$ is embedded in $H^\alpha_p(\mu)$. □

In what follows we consider the case $p = 2$, and investigate $H^\alpha_2(\mu)$ in more detail. Let $\alpha > 1$ and $f \in H^\alpha_2(\mu)$. We claim that $(-A)^{1/2} f \in H^{\alpha-1}_2(\mu)$. In fact, writing $f = J^\alpha_\mu \varphi$ for some $\varphi \in L_2(\mu)$, we see from Proposition 2.4 that

$$(-A)^{1/2} f = (I - A)^{-\frac{\alpha-1}{2}} \widetilde{\varphi},$$

where $\widetilde{\varphi} = (I - A)^{-1/2}(-A)^{1/2}\varphi$, by using the operational calculus (cf [27, p.343-345]). Note that $\widetilde{\varphi} \in L_2(\mu)$ for $\varphi \in L_2(\mu)$. Thus $(-A)^{1/2} f = J^{\alpha-1}_\mu \widetilde{\varphi} \in H^{\alpha-1}_2(\mu)$. Conversely, if $(-A)^{1/2} f \in H^{\alpha-1}_2(\mu)$ $(\alpha > 1)$ then $f \in H^\alpha_2(\mu)$ in a similar way. The Riesz fractional derivative $(-A)^{1/2}$ behaves like a pseudo-differential operator of order 1, exactly the same as the classical case. On the other hand, if we instead consider $L^p_s(X)$ introduced by Strichartz as above, then we see that $(-A)^{1/2} f \in L^2_{s-w/2}(X)$ $(s > w/2)$ if and only if $f \in L^2_s(X)$, and $(-A)^{1/2}$ behaves like a pseudo-differential operator of order $w/2$.

Using the spectral resolution (cf. Proposition 2.4), we see that for any $\alpha > 0$,

$$H^\alpha_2(\mu) = \left\{ f \in L_2(\mu) : \int_0^\infty (1 + \lambda)^\alpha d(E_\lambda f, f) < \infty \right\} \tag{3.4}$$

For $0 < \alpha \le 1$, we give a simple characterization of $H^\alpha_2(\mu)$. To do this, we introduce a function $j_\alpha : X \times X \to \mathbb{R}$ for $\alpha > 0$ by $j_\alpha(x, y) = 0$ if $x = y$, and

$$j_\alpha(x, y) = \frac{1}{\chi(\alpha, 1)} \int_0^\infty t^{-\alpha-1} e^{-t} G(t, x, y) dt \tag{3.5}$$

if $x \ne y$. Define a functional W_α by

$$W_\alpha(f) = \int_X \int_X (f(x) - f(y))^2 j_\alpha(x, y) d\mu(y) d\mu(x) \tag{3.6}$$

for $f \in L_2(\mu)$.

Theorem 3.3. *Let $H^\alpha_2(\mu)$ be defined as above. Then, if $0 < \alpha < 1$,*

$$H^\alpha_2(\mu) = \left\{ f \in L_2(\mu) : W_\alpha(f) < \infty \right\} \tag{3.7}$$

with $\|f\|_{H^\alpha_2(\mu)} \cong (\|f\|_2^2 + W_\alpha(f))^{1/2}$. If $\alpha = 1$,

$$H^1_2(\mu) = \left\{ f \in L_2(\mu) : \mathcal{E}(f, f) < \infty \right\} \tag{3.8}$$

with $\|f\|_{H^1_2(\mu)} \cong (\|f\|_2^2 + \mathcal{E}(f, f))^{1/2}$, where $\mathcal{E}$ is defined as (2.13).

Proof. Let $0 < \alpha < 1$. Let $f \in H^\alpha_2(\mu)$, and write $f = J^\alpha_\mu \varphi$ for some $\varphi \in L_2(\mu)$. Then

$$\varphi = D^\alpha f = (I - A)^{\alpha/2} f\ .$$

Therefore,

$$
\begin{aligned}
\|f\|^2_{H_2^\alpha(\mu)} &= \|\varphi\|_2^2 = \Big((I-A)^{\alpha/2}f,(I-A)^{\alpha/2}f\Big) = (f,(I-A)^\alpha f)\\
&= (f, D^{2\alpha} f) = \frac{1}{\chi(\alpha,1)}\int_0^\infty t^{-\alpha-1}(f, f-e^{-t}G_t f)dt\\
&= \frac{1}{\chi(\alpha,1)}\int_0^\infty t^{-\alpha-1}\Big[(1-e^{-t})\|f\|_2^2 + e^{-t}(f,f-G_tf)\Big]dt\\
&= \|f\|_2^2 + \frac{1}{2\chi(\alpha,1)}\int_0^\infty t^{-\alpha-1}e^{-t}\Big[\int_X\int_X (f(y)-f(x))^2 G(t,x,y)d\mu(y)d\mu(x)\Big]dt\\
&= \|f\|_2^2 + \frac{1}{2}\int_X\int_X (f(y)-f(x))^2 j_\alpha(x,y)d\mu(y)d\mu(x),
\end{aligned}
$$

proving (3.7).
If $\alpha = 1$, we see that

$$
\begin{aligned}
\|f\|^2_{H_2^\alpha(\mu)} &= \|\varphi\|_2^2 = ((I-A)^{1/2}f,(I-A)^{1/2}f)\\
&= \int_0^\infty (1+\lambda)d(E_\lambda f, f) = \|f\|_2^2 + \mathcal{E}(f,f),
\end{aligned}
$$

and so (3.8) holds. □

Corollary 3.4. *Suppose that (X,ρ,μ) is a connected metric measure space, endowed with a local, stochastically complete heat kernel $G(t,x,y)$ satisfying (G7) with*

$$\int_0^\infty s^{d+w-1}\Phi_2(s)ds < \infty. \tag{3.9}$$

Then $H_2^\alpha(\mu) = H_2^\alpha(d,w)$ with equivalent norms, where

$$H_2^\alpha(d,w) = \Big\{f\in L_2(\mu) : \int_X\int_{\rho(y,x)\le 1}\frac{(f(y)-f(x))^2}{\rho(x,y)^{d+\alpha w}}d\mu(y)d\mu(x)<\infty\Big\} \tag{3.10}$$

if $0<\alpha<1$, whilst

$$H_2^1(d,w) = \Big\{f\in L_2(\mu) : \sup_{0<r<1} r^{-(d+w)}\int_X\int_{B(x,r)}(f(y)-f(x))^2d\mu(y)d\mu(x)<\infty\Big\} \tag{3.11}$$

if $\alpha=1$. The norm of $f\in H_2^\alpha(d,w)$ is defined in an obvious way for $\alpha>0$.

Proof. Case $0<\alpha<1$. Since $G(t,x,y)$ satisfies $(G7)$ with (3.9), we can obtain that $j_\alpha(x,y)\le C\,\rho(x,y)^{-(d+\alpha w)}$ for $x,y\in X (x\neq y)$, and

$$j_\alpha(x,y)\ge C^{-1}\rho(x,y)^{-(d+\alpha w)} \tag{3.12}$$

if $0<\rho(x,y)\le 1$, in a similar way to (2.37) whether X is bounded or unbounded, where C is independent of x,y. Thus (3.10) follows from (3.7). For the case $\alpha=1$, we have that

$$\mathcal{E}(f,f)\cong \sup_{0<r<1} r^{-(d+w)}\int_X\int_{B(x,r)}(f(y)-f(x))^2d\mu(y)d\mu(x)$$

by virtue of (G7) and (3.9), see the details in [12, Theorem 4.2, p.2076] (note that μ is a d-measure if (G7) holds with (3.9), see Proposition 2.6). Thus (3.11) follows from (3.8). □

Remark. If X is a subset of $\mathbb{R}^n$ supporting a d-measure, the spaces $H_2^\alpha(d,w)$ for $0<\alpha\le 1$ and $w=2$ were introduced by Jonsson and Wallin in [16, 17]; in particular, they are the traces of the Sobolev spaces $H_2^{\alpha+\frac{n-d}{2}}(\mathbb{R}^n)$ on X for $0<\alpha<1$.

4. Embedding theorems

Assume that $G(t,x,y)$ satisfies $(G1)$-$(G7)$ with (3.9). In this section we discuss the embedding theorems for $H_p^\alpha(\mu)$ for $\alpha>0$ and $1<p<\infty$.

Theorem 4.1. *Let $H_p^\alpha(\mu)$ be defined as in Definition 3.1 for $\alpha>0$ and $1<p<\infty$. Then*

(1) *if $d>\alpha p(w/2)$, then $H_p^\alpha(\mu)$ embeds in $L^q(\mu)$ $\quad (q=\frac{pd}{d-\alpha p(w/2)})$;*
(2) *if $d=\alpha p(w/2)$, then $H_p^\alpha(\mu)$ embeds in $L^q(\mu)$ $\quad (1<q<\infty)$.*

Proof. (1) Case $d>\alpha p(w/2)$. Let $f\in H_p^\alpha(\mu)$, and write $f=J_\mu^\alpha\varphi, \varphi\in L_p(\mu)$. Since $0<\alpha<\frac{2d}{pw}<\frac{2}{w}d$, we see from (2.37) that there exists some $C>0$ independent of x and f such that

$$|f(x)|\le C\int_X \rho(x,y)^{-(d-\alpha w/2)}|\varphi(y)|\,d\mu(y), \tag{4.1}$$

which implies that $f\in L^q(\mu)$ if $d>\alpha p(w/2)$. The proof for this is standard, see for example [14, p.20-21]. For the reader's convenience, we sketch the arguments. Write

$$\begin{aligned}\int_X \rho(x,y)^{-(d-\alpha w/2)}|\varphi(y)|\,d\mu(y) &= \int_{\rho(y,x)>\delta}\rho(x,y)^{-(d-\alpha w/2)}|\varphi(y)|\,d\mu(y)\\ &\quad+\int_{\rho(y,x)\le\delta}\rho(x,y)^{-(d-\alpha w/2)}|\varphi(y)|\,d\mu(y)\\ &:= g_\delta(x)+b_\delta(x). \end{aligned}\tag{4.2}$$

Using Hölder's inequality, we have that

$$\begin{aligned}g_\delta(x)&\le\|\varphi\|_p\left(\int_{\rho(x,y)>\delta}\rho(x,y)^{-p'(d-\alpha w/2)}d\mu(y)\right)^{1/p'}\\ &\le C\,\|\varphi\|_p\delta^{-(d/p-\alpha w/2)},\end{aligned}\tag{4.3}$$

where C is independent of φ,δ and x, and $p'=\frac{p}{p-1}$. Here we have used the fact that

$$\begin{aligned}\int_{\rho(x,y)>\delta}\rho(x,y)^{-p'(d-\alpha w/2)}d\mu(y)&=\sum_{k=1}^\infty\int_{2^k\delta<\rho(x,y)\le 2^{k+1}\delta}\rho(x,y)^{-p'(d-\alpha w/2)}d\mu(y)\\ &\le\sum_{k=1}^\infty(2^k\delta)^{-p'(d-\alpha w/2)}\mu(B(x,2^{k+1}\delta))\\ &\le\sum_{k=1}^\infty(2^k\delta)^{-p'(d-\alpha w/2)}C(2^{k+1}\delta)^d \quad \text{(by (2.33))}\\ &=C\delta^{d-p'(d-\alpha w/2)}\sum_{k=1}^\infty 2^{k(d-p'(d-\alpha w/2))}\\ &\le C\delta^{d-p'(d-\alpha w/2)} \quad (\text{since } d-p'(d-\alpha w/2)<0).\end{aligned}$$

On the other hand,

$$
\begin{aligned}
b_\delta(x) &= \int_{\rho(x,y)\le\delta} \rho(x,y)^{-(d-\alpha w/2)}|\varphi(y)|\,d\mu(y)\\
&= \sum_{k=0}^{\infty}\int_{2^{-(k+1)}\delta<\rho(x,y)\le 2^{-k}\delta} \rho(x,y)^{-(d-\alpha w/2)}|\varphi(y)|\,d\mu(y)\\
&\le \sum_{k=0}^{\infty}(2^{-(k+1)}\delta)^{-(d-\alpha w/2)}\int_{\rho(x,y)\le 2^{-k}\delta}|\varphi(y)|d\mu(y) \qquad (4.4)\\
&\le \sum_{k=0}^{\infty}(2^{-(k+1)}\delta)^{-(d-\alpha w/2)}\mu(B(x,2^{-k}\delta))M_\mu(\varphi)(x)\\
&\le C\,\delta^{\frac{\alpha w}{2}}M_\mu(\varphi)(x),
\end{aligned}
$$

where

$$M_\mu\varphi(x) := \sup_{0<r\le r_0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|\varphi(y)|d\mu(y)$$

is the maximal function of φ satisfying

$$\|M_\mu\varphi\|_p \le C\,\|\varphi\|_p \quad \text{for all } \varphi\in L_p(\mu). \tag{4.5}$$

Combining (4.1)-(4.5), we see that

$$|f(x)| \le C\,\left(\delta^{-(d/p-\alpha w/2)}\,\|\varphi\|_p + \delta^{\frac{\alpha w}{2}}\,M_\mu\varphi(x)\right).$$

Minimizing the right-hand side, we have that

$$|f(x)| \le C\,\|\varphi\|_p^{\frac{\alpha wp}{2d}}\,(M_\mu\varphi(x))^{\frac{2d-\alpha wp}{2d}},$$

which combines with (4.5) to yield that

$$\|f\|_q \le C\,\|\varphi\|_p, \quad q=\frac{dp}{d-\alpha pw/2}.$$

Case (ii) $d=\alpha p/2w$ follows from (i) in a standard way. We omit the details. □

We give the embedding of $H_p^\alpha(\mu)$ for $p=2$, if $d<\alpha w$ and $0<\alpha\le 1$. For $0<\sigma\le 1$, let $C^\sigma(X)$ be the Hölder spaces on X, that is

$$C^\sigma(X)=\Big\{f\in C(X): \sup_{x,y\in X, x\neq y}\frac{|f(y)-f(x)|}{\rho(y,x)^\sigma}<\infty\Big\}.$$

The *norm* of $f\in C^\sigma(X)$ is

$$\|f\|_{C^\sigma(X)} = \|f\|_C(X) + \sup_{x,y\in X,x\neq y}\frac{|f(y)-f(x)|}{\rho(y,x)^\sigma}.$$

Theorem 4.2. *If $d<\alpha w$ and $0<\alpha\le 1$, then $H_2^\alpha(\mu)$ embeds in $C^\sigma(X)$, where $\sigma=\frac{\alpha w-d}{2}$.*

Proof. Let $d<\alpha w$ and $0<\alpha\le 1$. By Corollary 3.4, it suffices to show that $H_2^\alpha(d,w)$ embeds in $C^\sigma(X)$ with $\sigma=\frac{\alpha w-d}{2}$. But this is proved in Theorem 4.1 (iii) of [12] if $\alpha=1$, and in [15] for $0<\alpha<1$ (Note that the chain condition implies that $\alpha w\le d+2$ (cf.[12]), and so $\sigma\le 1$). □

By (3.2) and Theorem 4.2 we have that $L_s^2(X)$ embeds in $C^\sigma(X)$ with $\sigma=s-\frac{d}{2}$ if $\frac{d}{2}<s\le\frac{w}{2}$, see the same result in [22].

5. Besov spaces

Let (X,ρ,μ) be a connected metric measure space endowed with a $G(t,x,y)$ satisfying $(G1)$-$(G7)$. Let $\{G_t\}_{t\geq 0}$ be the semigroup associated with $G(t,x,y)$ as in (2.1). In this section we define various Besov spaces $B^{\alpha}_{p,q}(\mu)$ for $\alpha\in\mathbb{R}$ and $1\leq p,q\leq\infty$.

Definition 5.1. Let $1\leq p,q\leq\infty$, and let $\alpha\in\mathbb{R}$. Define $B^{\alpha}_{p,q}(\mu)$ as follows

(1) $\alpha>0, k=[\frac{\alpha}{2}]+1, B^{\alpha}_{p,q}(\mu)=\{f\in L_p(\mu):(\int_0^\infty(t^{k-\frac{\alpha}{2}}\|\frac{\partial^k}{\partial t^k}G_tf\|_p)^q\frac{dt}{t})^{\frac{1}{q}}<\infty\}$;

(2) $\alpha=0, k=1, B^{\alpha}_{p,q}(\mu)=\{f\in L_p(\mu):(\int_0^\infty(t\,\|\frac{\partial}{\partial t}G_tf\|_p)^q\frac{dt}{t})^{\frac{1}{q}}<\infty\}$;

(3) $\alpha<0, k=0, B^{\alpha}_{p,q}(\mu)=\{f\in L_p(\mu):(\int_0^\infty(t^{-\frac{\alpha}{2}}\|G_tf\|_p)^q\frac{dt}{t})^{\frac{1}{q}}<\infty\}$

with the obvious norm (the integrals above are clearly modified if $q=\infty$).

The Besov spaces as above for the classical case were given in [9] by using the Gauss-Weierstrass heat kernel. For other approaches to the fractal case, see [8, 22, 23, 25].

Theorem 5.2. *For $\alpha\geq 0$, we have that $B^{\alpha}_{2,2}(\mu)=H^{\alpha}_2(\mu)$ with equivalent norms.*

Proof. Let $\alpha\geq 0$. Note that

$$H^{\alpha}_2(\mu)=\Big\{f\in L_2(\mu):\int_0^\infty(1+\lambda)^{\alpha}d(E_\lambda f,f)<\infty\Big\}\quad(\alpha\geq 0).$$

On the other hand, we have that for any integer $k\geq\left[\frac{\alpha}{2}\right]+1$,

$$\Big\|\frac{\partial^k}{\partial t^k}G_tf\Big\|_2^2=\int_0^\infty\lambda^{2k}e^{-2\lambda t}d(E_\lambda f,f),$$

and so

$$\begin{aligned}\int_0^\infty t^{2k-\alpha-1}\Big\|\frac{\partial}{\partial t}G_tf\Big\|_2^2dt&=\int_0^\infty t^{2k-\alpha-1}\Big(\int_0^\infty\lambda^{2k}e^{-2\lambda t}d(E_\lambda f,f)\Big)dt\\&=\int_0^\infty\lambda^{2k}d(E_\lambda f,f)\Big(\int_0^\infty t^{2k-\alpha-1}e^{-2\lambda t}dt\Big)\\&=2^{\alpha-2k}\Gamma(2k-\alpha)\int_0^\infty\lambda^{\alpha}d(E_\lambda f,f),\end{aligned}$$

proving the theorem. □

6. Discussions

Starting from the existence of a local, stochastically complete heat kernel $G(t,x,y)$ with the two-sided estimate (G7) on a connected metric measure space (X,ρ,μ), we have obtained various Besov spaces; in particular, we have at hand the (fractional) Sobolev spaces $H^{\alpha}_2(d,w)(0<\alpha\leq 1)$, see (3.10) and (3.11). These spaces contain two parameters, d and w, which are the *Hausdorff dimension* of X and the *walk dimension* of the diffusion on (X,ρ,μ) respectively. The key point is that these spaces are *dense* in $L_2(\mu)$. There is a natural question: is it possible to obtain the existence of a heat kernel satisfying a two-sided estimate (G7), if there exists the function space, say $H^{\alpha}_2(d,w)$ (cf (3.10) or (3.11)) for some $0<\alpha\leq 1$ that is *dense* in $L_2(\mu)$. This question has been answered for $0<\alpha<1$ and $w=2$. Chen and Kumagai [7] applied the probability approach to show that, if X is a d-subset of $\mathbb{R}^n (n \geq 1)$ and μ is a d-measure, then there is a heat kernel $p^{(\sigma)}(t,x,y)$ satisfying

$$C^{-1}\, t^{-\frac{d}{2\alpha}} \Phi\big(t^{-1/(2\alpha)}|x-y|\big) \leq p^{(\sigma)}(t,x,y) \leq C\, t^{-\frac{d}{2\alpha}} \Phi\big(t^{-1/(2\alpha)}|x-y|\big) \tag{6.1}$$

for all $0 < t < 1$ and all $x, y \in X$, where $0 < \alpha < 1$ and

$$\Phi(s) = (1+s)^{-(d+2\alpha)} \quad (s \geq 0).$$

(Note that there is no problem of the denseness property for $H_2^\alpha(d,w)$ if $0 < \alpha \leq 1$ and $w = 2$.) For each $0 < \alpha < 1$, the $p^{(\sigma)}(t,x,y)$ is not local, and the corresponding process is a jump process on X. The limiting case $\alpha = 1$ is still open. Recently, in [28] a regular local Dirichlet form has been constructed on any bounded d-set which corresponds to the limiting case $\alpha = 1$. All the results mentioned above are treated only when $w = 2$. It would be interesting to investigate the general case $w > 2$, and moreover, to show the equivalence between the existence of a certain class of potential spaces and the existence of heat kernels with two-sided decay estimates.

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