RIESZ POTENTIALS AND LIOUVILLE OPERATORS ON FRACTALS

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Abstract

An analogue to the theory of Riesz potentials and Liouville operators in $\mathbb{R}^n$ for arbitrary fractal d-sets is developed. Corresponding function spaces agree with traces of euclidean Besov spaces on fractals. By means of associated quadratic forms we construct strongly continuous semigroups with Liouville operators as infinitesimal generator. The case of Dirichlet forms is discussed separately. As an example of related pseudodifferential equations the fractional heat-type equation is solved.

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0 Introduction

In [17] the Riesz potential of order s of a fractal d-measure μ in $\mathbb{R}^n$ with compact support Γ is defined as

$$I_\mu^s f(x) := \text{const} \int |x-y|^{-(d-s)} f(y)\mu(dy) \ ,$$

$f \in L_2(\mu)$, where $0 < s < d < n$. (Examples for the measure μ are the Hausdorff measures $\mathcal{H}^d$ on arbitrary self-similar sets Γ of dimension d.) Such potentials have a long tradition for the case, where μ is replaced by the Lebesgue measure. For more general μ only a view properties are known. Some references may be found in [17]. We also refer to the fundamental paper [2]. In connection with harmonic analysis and stochastic processes on fractals these potentials appear in a new light.
The aim of the present paper is to continue this study in order to get a deeper insight into the interplay between geometry of and analysis on fractal sets and the corresponding properties of the embedding euclidean space. By means of euclidean charts of metric spaces as introduced in [16] these relationships might pay a role also for more general metric spaces. Some basic properties as compactness of I_μ^s are proved in [17] for general metric spaces. The whole spectral properties are extended in [16] to the case of some equivalent quasi-metrics in arbitrary quasi-metric spaces.
Here we focus our attention on pseudodifferential operators, quadratic forms and generated semigroups associated with the above Riesz potentials.
In Section 1 we summarize some classical euclidean results which are important for our purposes. This concerns euclidean Riesz potentials, Liouville operators, Fourier representation and related function spaces.
Section 2 contains a brief survey on the properties of I_μ^s proved in [17].
In Section 3 the method of traces of Besov spaces on fractals is described as it has been introduced by H. Triebel. (This is an equivalent approach to the fractal Besov spaces defined by Jonsson and Wallin.) We concentrate on the Hilbert spaces $H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n)$ and $H^{\frac{s}{2}}(\Gamma)$ and prove an isometry property of the operator $\sqrt{I_\mu^s}$ (Theorem 3.1).
The fractal Liouville operators D_μ^s defined as the inverses of the Riesz potentials are studied in Section 4. They induce quadratic forms

$$\mathcal{E}_\mu^s(f,g) := \langle \sqrt{D_\mu^s}\ f, \sqrt{D_\mu^s}\ g\rangle_{L_2(\mu)}$$

on $L_2(\mu)$. In Theorem 4.1, the first main result of the paper, we show that $\mathcal{E}_\mu^s$ is a regular closed quadratic form on $L_2(\mu)$ with domain $H^{\frac{s}{2}}(\Gamma)$. Furthermore, it is the trace of its euclidean counterpart

$$\mathcal{E}^{s+n-d}(f,g) = \langle D^{\frac{s}{2}+\frac{n-d}{2}} f, D^{\frac{s}{2}+\frac{n-d}{2}} g\rangle_{L_2(R^n)}$$

for the euclidean Liouville operator $D^{\frac{s}{2}+\frac{n-d}{2}}$ with domain $H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n)$ and a certain extension.
In Section 5 the strongly continuous semigroups $(T_t)_{t\geq 0}$ generated by $\mathcal{E}_\mu^s$ are considered. As an application we construct the unique solution of the fractal pseudodifferential equation

$$\frac{\partial u}{\partial t}(t,\cdot) = -D_\mu^s\ u(t,\cdot)\ ,\quad t \geq 0\ ,$$

with initial condition $u(0,\cdot) = f \in L_2(\mu)$.
Finally we show in Section 6 that for $0 < s < d \leq n$ with $s + n - d \leq 2$ the quadratic form $\mathcal{E}_\mu^s$ is a regular Dirichlet form. This leads to associated Markov processes on the fractal set Γ .
All results in Sections 1–4 are related to both the complex and real valued cases. The corresponding assertions from [17] extend immediately to the complex case. (The same holds true for the marginal dimension $d = n$, where the corresponding theory is classical.) In Sections 5 and 6 we restrict to real function spaces.

1 Survey on related euclidean results from potential theory, Fourier analysis and stochastic processes

Let $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ be the classical **Laplace operator** in $\mathbb{R}^n$. The inverse operator of $-\Delta$ is given by the Newtonian potential

$$I^2 f(x) := \text{const} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}\ f(y) dy$$

for $f \in L_2(\mathbb{R}^n)$ if $n > 2$. The Riesz potential of order 1

$$I^1 f(x) := const \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}}\ f(y)\ dy$$

corresponds to $(-\Delta)^{1/2}$. In general, the **Riesz potential** of order $0 < \sigma < n$

$$I^\sigma f(x) := \text{const} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\sigma}}\ f(y)\ dy\ ,$$

$f \in L_2(\mathbb{R}^n)$, is the inverse of the **pseudodifferential operator** of order σ

$$D^\sigma := (-\Delta)^{\sigma/2}$$

which is also called **Liouville operator**.
The constant is equal to

$$c_n(\sigma) := \Gamma\left(\frac{n-\sigma}{2}\right) / (2^\sigma \pi^{n/2} \Gamma\left(\frac{\sigma}{2}\right))$$

and

$$G^\sigma(z) := c_n(\sigma)\ |z|^{-(n-\sigma)}$$

is called **Riesz kernel** of order σ in $\mathbb{R}^n$. Up to certain exceptional orders σ the operators D^σ can be realized by hypersingular integrals (cf. Stein [12] or Samko, Kilbas and Marichev [11]).

In the Fourier analytical approach for arbitrary $\sigma \geq 0$,

$$\begin{aligned} D^\sigma &:= F^{-1}(|\xi|^\sigma\ F) \\ I^\sigma &:= F^{-1}(|\xi|^{-\sigma}\ F) \end{aligned}$$

are defined on distributions, where F means the Fourier transform in $\mathbb{R}^n$. Similarly, fractional powers of the **resolvent** $(Id - \Delta)^{-1}$ are introduced:

$$\mathcal{D}^\sigma := F^{-1}((1+|\xi|^2)^{\sigma/2} F)$$

with inverse operators

$$\mathcal{J}^\sigma := F^{-1}((1+|\xi|^2)^{-\sigma/2} F)$$

i. e.,

$$\mathcal{J}^\sigma = (Id - \Delta)^{\sigma/2}\ .$$

The latter are representable as

$$\mathcal{J}^\sigma f(x) := \int_{\mathbb{R}^n} G_B^\sigma(x-y)\ f(y)\ dy$$

for the **Bessel kernel** G_B^σ which has a better behavior at infinity than the Riesz kernel G^σ. For $\sigma \geq 0$ one also denotes

$$\mathcal{J}^{-\sigma} := \mathcal{D}^\sigma\ .$$

Related distribution spaces are

$$H^\sigma(\mathbb{R}^n) := \mathcal{J}^\sigma(L_2(\mathbb{R}^n))\ , \quad \sigma \in \mathbb{R}\ ,$$

which may be provided with a Hilbert space structure by

$$\begin{aligned} \langle f, g\rangle_{H^\sigma(\mathbb{R}^n)} &:= \int_{\mathbb{R}^n} (1+|\xi|^2)^\sigma\ F(f)(\xi)\ \overline{F(g)(\xi)}\ d\xi \\ &= \langle \mathcal{D}^\sigma f, \mathcal{D}^\sigma g\rangle_{L_2(\mathbb{R}^n)}. \end{aligned}$$

Up to (semi) norm equivalence the **space of Bessel potentials** $H^\sigma(\mathbb{R}^n)$ agrees with the Sobolev-Slobodeckij space $W_2^\sigma(\mathbb{R}^n)$ (for $\sigma \geq 0$) and with the Besov space $B_{2,2}^\sigma(\mathbb{R}^n)$. Moreover, for $\sigma > 0$ the space $H^\sigma(\mathbb{R}^n)$ consists of those functions $f \in L_2(\mathbb{R}^n)$ which possess fractional derivatives $D^\sigma f$ (in the first sense) in $L_2(\mathbb{R}^n)$. (General references are, e. g. [12], [15], [11] and the literature cited there.)
We also consider the quadratic forms

$$\begin{aligned} \mathcal{E}^\sigma(f,g) &:= \int_{\mathbb{R}^n} |\xi|^\sigma \; F(f)(\xi) \; \overline{F(g)(\xi)} \; d\xi \\ &= \langle D^{\sigma/2} f, D^{\sigma/2} g \rangle_{L_2(\mathbb{R}^n)} \end{aligned}$$

in $L_2(\mathbb{R}^n)$ with domain $H^{\sigma/2}(\mathbb{R}^n)$, which are closed and regular. Therefore the operator $-D^\sigma$ related to this form is the infinitesimal generator of a strongly continuous contraction semigroup in $\mathbb{R}^n$. For $0 < \sigma \leq 2$ this corresponds to the **Dirichlet form** and the **Markov semigroup** of a σ-stable symmetric **Levy process** in $\mathbb{R}^n$ (with characteristic function $exp(-t|\xi|^\sigma)$). (More details on the last notions and results may be found in [3] and [5].)

2 Riesz potentials of fractal d-measures in $\mathbb{R}^n$

The Lebesgue measure as reference measure is now replaced by a **finite Borel measure** μ in $\mathbb{R}^n$ with **compact support** Γ. We suppose that for all $x \in \Gamma$ and $0 < r \leq 1$,

$$C^{-1} \leq \frac{\mu(B(x,r))}{r^d} \leq C$$

for some constant $C > 0$, where $B(x,r)$ denotes the closed ball with centre x and radius r. Such measures are called d**-measures** with scaling exponent $d \in (0,n]$. In [17] the **fractal Riesz potentials** of order $s < d$ are studied:

$$I_\mu^s f(x) := c_n(s+n-d) \int \frac{1}{|x-y|^{d-s}} \; f(y) \; d\mu(y) \; ,$$

$f \in L_2(\mu)$. They are shown to be closely related to their euclidean counterpart by

$$I_\mu^s f = tr_\Gamma \circ I^\sigma \circ (f\mu)$$

for $\sigma := s+n-d$, where $n-d$ appears as a fractal defect. $f\mu$ stands for the distribution given by integrating with respect to $fd\mu$ and the trace operator tr_Γ agrees here with the restriction to Γ. (Its general construction going back to Triebel [14] will be summarized below.) In [17] it is shown (for the case of general metric spaces) that I_μ^s is a compact self-adjoint operator in the Hilbert space $L_2(\mu)$ and in the case of $\mathbb{R}^n$ it is positive. The method of proving the last fact (Theorem 3.2 in [17]) is based on an isometry property which is essential for the aim of the present paper. Furthermore, the asymptotic behavior of the eigenvalues λ_k of I^s_μ in $\mathbb{R}^n$ is determined by

$$\lambda_k \asymp k^{-s/d} \ , \quad k \longrightarrow \infty \ ,$$

with methods of [14] (Theorem 3.3 in [17]). This corresponds to the Weyl spectrum for the fractional powers of the Laplace operator in the classical euclidean case. In Triebel, Yang [16] these asymptotics are extended to d-measures in rather general quasi-metric spaces.

3 Traces and Besov spaces on fractals

Traces of euclidean Besov spaces to fractal d-sets Γ have first been introduced by Johnson, Wallin [6] and studied in subsequent papers. We use here the modified approach of Triebel [14], which extends the notion of euclidean tracing in the theory of function spaces to the fractal case. (For $0 < s < d \leq 1$ both versions agree.) The main ideas are as follows: (We need only the case $p = 2$ from [14].) The trace operator tr_Γ for Schwartz functions φ is defined by the restriction to Γ. As a first basic result the norm estimate

$$||\varphi| \ L_2(\mu)|| \leq \text{const} \, ||\varphi| \ B_{2,1}^{\frac{n-d}{2}}(\mathbb{R}^n)||$$

is derived by means of atomic decompositions. Then tr_Γ is continuously extended to an operator from $B_{2,1}^{\frac{n-d}{2}}(\mathbb{R}^n)$ into $L_2(\mu)$. In a second step the image is shown to be the whole $L_2(\mu)$, i.e.,

$$tr_\Gamma \left(B_{2,1}^{\frac{n-d}{2}}(\mathbb{R}^n) \right) = L_2(\mu) \ .$$

Finally, **fractal Besov spaces** are introduced by

$$B^s_{2,q}(\Gamma) := tr_\Gamma \left(B_{2,q}^{s+\frac{n-d}{2}}(\mathbb{R}^n) \right)$$

for $s > 0$ and $0 < q \leq \infty$. They become quasi-Banach spaces (Banach spaces if $q \geq 1$) by

$$||f| \ B^s_{2,q}(\Gamma)|| := \inf ||g| \ B_{2,q}^{s+\frac{n-d}{2}}(\mathbb{R}^n)||$$

where the infimum is taken over all $g \in B_{2,q}^{s+\frac{n-d}{s}}(\mathbb{R}^n)$ with $tr_\Gamma g = f$. Similarly,

$$H^s(\Gamma) := tr_\Gamma \left(H^{s+\frac{n-d}{2}}(\mathbb{R}^n) \right)$$

and the Hilbert space structure of $H^{s+\frac{n-d}{2}}(\mathbb{R}^n)$ generates that of $H^s(\Gamma)$. Moreover,

$$H^{s+\frac{n-d}{2}}(\mathbb{R}^n) = \left\{ g \in H^{s+\frac{n-d}{2}}(\mathbb{R}^n) : \ tr_\Gamma g = 0 \right\} \oplus H^s(\Gamma)$$

(cf. [14], 25.1).
For our purposes we slightly modify the construction. In $H^\sigma(\mathbb{R}^n)$ an equivalent Hilbert space structure is given when using a modified Riesz-Bessel kernel G_R^σ instead of the Bessel kernel G_B^σ. It is equivalent to the Bessel kernel at infinity and for the points of Γ it is defined by the Riesz kernel, í.e.,

$$G_R^\sigma(z) := G^\sigma(z) \quad \text{if} \quad |z| < R \ ,$$

where $R > 2\ \mathrm{diam}(\Gamma)$ (cf. [17]).
We write $I_R^\sigma f(x) := \int G_R^\sigma(x-y)f(y)dy$, $D_R^\sigma := (I_R^\sigma)^{-1}$, and $||f|\ H_R^\sigma(\mathbb{R}^n)||$, $\langle f,g\rangle_{H_R^\sigma(\mathbb{R}^n)}$ for the equivalent norm and scalar product in $H^\sigma(\mathbb{R}^n)$. Furthermore, in the space $H^s(\Gamma)$ introduced above we use the equivalent norm

$$||f|\ H_R^s(\Gamma)|| := \inf ||g|\ H_R^{s+\frac{n-d}{2}}(\mathbb{R}^n)||$$

where the infimum is taken over all $g \in H^{s+\frac{n-d}{2}}(\mathbb{R}^n)$ with $tr_\Gamma g = f$. The corresponding scalar product is determined by

$$\begin{aligned} 4\langle f,g\rangle_{H_R^s(\Gamma)} &:= ||f+g|\ H_R^s(\Gamma)||^2 - ||f-g|\ H_R^s(\Gamma)||^2 \\ &+ \ i||f+ig|\ H_R^s(\Gamma)||^2 - i||f-ig|\ H_R^s(\Gamma)||^2 \ . \end{aligned}$$

$H^{s+\frac{n-d}{2}}(\mathbb{R}^n)$ and $H^s(\Gamma)$ provided with these structures are denoted by $H_R^{s+\frac{n-d}{2}}(\mathbb{R}^n)$ and $H_R^s(\Gamma)$, resp. (Below we will show that the norm in $H_R^s(\Gamma)$ does not depend on the radius R.) The analogue of the above Hilbert space decomposition reads:

$$H_R^{s+\frac{n-d}{2}}(\mathbb{R}^n) = \left\{ g \in H_R^{s+\frac{n-d}{2}}(\mathbb{R}^n) :\ tr_\Gamma g = 0 \right\} \oplus H_R^s(\Gamma) \ . \tag{3.1}$$

The operator $\sqrt{I_\mu^s}$ plays a basic role for the Hilbert space $H_R^{s/2}(\Gamma)$.

3.1 Theorem. *$\sqrt{I_\mu^s}$ is an isometry from $L_2(\mu)$ onto $H^{s/2}(\Gamma)$, i.e.,*

$$\langle \sqrt{I_\mu^s} f, \sqrt{I_\mu^s} g\rangle_{H_R^{s/2}(\Gamma)} = \langle f,g\rangle_{L_2(\mu)} \ .$$

Proof. In the proof of Theorem 3.3 in [17] it is shown that I_μ^s maps $L_2(\mu)$ into $H_R^{s/2}(\Gamma)$,

$$\langle f,g\rangle_{L_2(\mu)} = \langle I_\mu^s f, g\rangle_{H_R^{s/2}(\Gamma)}$$

for $f,g \in H_R^{s/2}(\Gamma)$, and I_μ^s is a compact self-adjoint operator on the Hilbert space $H_R^{s/2}(\Gamma)$. Moreover, by the above equality it is positive. Therefore we obtain for any $f,g \in H_R^{s/2}(\Gamma)$

$$\langle f,g\rangle_{L_2(\mu)} = \langle \sqrt{I_\mu^s}\ f, \sqrt{I_\mu^s}\ g\rangle_{H_R^{s/2}(\Gamma)} \ .$$

It remains to extend this isometry to all $f, g \in L_2(\mu)$ and to show that

$$\sqrt{I_\mu^s}(L_2(\mu)) = H_R^{s/2}(\Gamma) \ .$$

The operator I_μ^s is also compact, selfadjoint and positive in the Hilbert space $L_2(\mu)$ (see [17]). From the general theory of such operators it follows that there exist complete orthogonal sequences of eigenvectors of the operator I_μ^s in $L_2(\mu)$ as well as in $H_R^{s/2}(\Gamma)$. By construction, these systems may be chosen the same. Let now f be an arbitrary function from $L_2(\mu)$. Then f may be approximated in $L_2(\mu)$ by linear combinations f_n of the eigenvectors $e_1, e_2, \ldots$. The above equality implies

$$\langle e_i, e_i \rangle_{L_2(\mu)} = \langle \sqrt{I_\mu^s}\ e_i, \sqrt{I_\mu^s}\ e_i \rangle_{H_R^{s/2}(\Gamma)}$$

for any eigenvector e_i. Hence,

$$||f_n - f_m|\ L_2(\mu)|| = ||\sqrt{I_\mu^s} f_n - \sqrt{I_\mu^s} f_m|\ H_R^{s/2}(\Gamma)|| \ .$$

f_n converges to f in $L_2(\mu)$ as $n \to \infty$. Therefore $\sqrt{I_\mu^s}\ f_n$ tends to some φ in $H_R^{s/2}(\Gamma)$. On the other hand,

$$||h|\ L_2(\mu)|| \leq const||h|\ H_R^{s/2}(\Gamma)|| \tag{3.2}$$

for any $h \in H_R^{s/2}(\Gamma)$ (see [14], (25.9)). Thus, $\sqrt{I_\mu^s}\ f_n$ converges to φ in $L_2(\mu)$, too. Since $\sqrt{I_\mu^s}$ is bounded in $L_2(\mu)$ we get $\varphi = \sqrt{I_\mu^s}\ f$. From this we infer

$$||f|\ L_2(\mu)|| = ||\sqrt{I_\mu^s} f|\ H_R^{s/2}(\Gamma)||$$

for any $f \in L_2(\mu)$. The equality for the scalar products is a consequence. Finally, any φ from the Hilbert space $H_R^{s/2}(\Gamma)$ is the limit as $n \to \infty$ of

$$\sum_{i=1}^n \varphi_i e_i = \sqrt{I_\mu^s} \left(\sum_{i=1}^n \varphi_i \lambda_i^{-1} e_i \right)$$

where $\varphi_i := \langle \varphi, e_i \rangle_{H_R^{s/2}(\Gamma)}$ and $\sqrt{I_\mu^s} e_i = \lambda_i e_i$. From the above isometry property we obtain that $\sum_{i=1}^n \varphi_i \lambda_i^{-1} e_i$ converges in $L_2(\mu)$ to some function f. Hence, $\sqrt{I_\mu^s}\ f = \varphi$. This yields

$$\sqrt{I_\mu^s}(L_2(\mu)) = H_R^{s/2}(\Gamma) \ .$$

□

4 Fractal Liouville operators and quadratic forms associated with Riesz potentials

In analogy to the case euclidean Riesz potentials (cf. Section 1) we introduce **fractional pseudodifferential operators** with respect to the d-measure μ as above by

$$D^s_\mu = (I^s_\mu)^{-1}$$

for $0 < s < d \leq n$. D^s_μ is called **fractal Liouville operator of order *s***. The **space $L^s_2(\mu)$ of Riesz potentials** of order s may be equipped with the scalar product

$$\langle f, g\rangle_{L^s_2(\mu)} := \langle D^s_\mu f, D^s_\mu g\rangle_{L_2(\mu)} \ .$$

Obviously, this generates a Hilbert space structure. In this way $L^s_2(\mu)$ may be considered as **fractal analogue** of the fractional Sobolev space $L^\sigma_2(\mathbb{R})$ of $L_2(\mathbb{R}^n)$-functions with derivatives D^σ in $L_2(\mathbb{R}^n)$. Recall that the latter is equivalent to $H^\sigma(\mathbb{R}^n)$. In distinction to the euclidean case we have

$$D^{s/2}_\mu \circ D^{s/2}_\mu \neq D^s_\mu \ .$$

Therefore we additionally introduce the quadratic form

$$\mathcal{E}^s_\mu(f, g) := \left\langle \sqrt{D^s_\mu} f, \sqrt{D^s_\mu} g \right\rangle_{L_2(\mu)}$$

in $L_2(\mu)$. (For general notions used here and in the sequel see Fukushima, Oshima, Takeda [3].)

4.1 Theorem. *(i)*

$$\mathcal{E}^s_\mu(f, g) = \langle f, g\rangle_{H^{s/2}_R(\Gamma)} \ ,$$

$f, g \in H^{s/2}(\Gamma)$. In particular, the scalar product in $H^{s/2}_R(\Gamma)$ does not depend on the choice of the radius R.

(ii) *$\mathcal{E}^s_\mu$ is a closed and regular quadratic form in $L_2(\mu)$ with domain $H^{s/2}(\Gamma)$.*

(iii) *$\mathcal{E}^s_\mu$ is the trace of its euclidean extension:*

$$\mathcal{E}^s_\mu(f, g) = \mathcal{E}^{s+n-d}(\tilde{f}, \tilde{g}) = \langle D^{\frac{s}{2}+\frac{n-d}{2}} \tilde{f}, D^{\frac{s}{2}+\frac{n-d}{2}} \tilde{g}\rangle_{L_2(R^n)}$$

where the Riesz extension $\tilde{f}$ of $f \in H^{s/2}(\Gamma)$ is determined by $\tilde{f} = \mathrm{ext}^s_\mu f := I^{s+n-d}(\varphi\mu)$, if $f = I^s_\mu \varphi$, and by continuity w.r.t. the seminorm $(\mathcal{E}^{s+n-d}(\cdot, \cdot))^{1/2}$.

Proof. (i) is a consequence of the definition of $\mathcal{E}_\mu^s$ and Theorem 3.1. In particular,

$$\operatorname{dom} \mathcal{E}_\mu^s = H^{s/2}(\Gamma)$$

which is dense in $L_2(\mu)$.
Closedness of a quadratic form $\mathcal{E}$ in $L_2(\mu)$ means that the domain of $\mathcal{E}$ is complete w.r.t. the $\mathcal{E}_1$-norm $(\mathcal{E}(f,f) + \|f| \ L_2(\mu)\|^2)^{1/2}$. By (3.2) the latter in our case is equivalent to $\|f| \ H^{s/2}(\Gamma)\|$. Furthermore, by construction the restriction $C^\infty(\Gamma)$ of $C_0^\infty(\mathbb{R}^n)$ to the fractal d-set Γ is dense in the Hilbert space $H_R^{s/2}(\Gamma)$. It is also dense in the space of continuous functions with the uniform norm. Hence, $C^\infty(\Gamma)$ forms a core of the quadratic form $\mathcal{E}_\mu^s$, i.e., the latter is regular. Thus, (ii) is proved.
For (iii) we use Theorem 3.2 in [17] which implies

$$\left\langle \sqrt{I_\mu^s}\,\varphi, \sqrt{I_\mu^s}\,\psi \right\rangle_{L_2(\mu)} = \langle I^{\frac{s}{2}+\frac{n-d}{2}}(\varphi\mu), I^{\frac{s}{2}+\frac{n-d}{2}}(\psi\mu)\rangle_{L_2(\mathbb{R}^n)}$$

for any $\varphi, \psi \in L_2(\mu)$. Suppose now that $\tilde{f} = I^{s+n-d}(\varphi\mu)$, $\tilde{g} = I^{s+n-d}(\psi\mu)$, i.e., their traces on Γ are $f = I_\mu^s\varphi$ and $g = I_\mu^s\psi$, respectively. Then we infer

$$\begin{aligned}
\mathcal{E}_\mu^s(f,g) &= \left\langle \sqrt{D_\mu^s}\,f, \sqrt{D_\mu^s}\,g \right\rangle_{L_2(\mu)} \\
&= \left\langle \sqrt{I_\mu^s}\,\varphi,\ \sqrt{I_\mu^s}\,\psi \right\rangle_{L_2(\mu)} \\
&= \langle I^{\frac{s}{2}+\frac{n-d}{2}}(\varphi\mu), I^{\frac{s}{2}+\frac{n-d}{2}}(\psi\mu)\rangle_{L_2(\mathbb{R}^n)} \\
&= \langle D^{\frac{s}{2}+\frac{n-d}{2}}\tilde{f}, D^{\frac{s}{2}+\frac{n-d}{2}}\tilde{g}\rangle_{L_2(\mathbb{R}^n)} \\
&= \mathcal{E}^{s+n-d}(\tilde{f},\tilde{g}) \ .
\end{aligned}$$

We call $\tilde{f}$ the corresponding **Riesz extension** of f denoted by $\operatorname{ext}_\mu^s f$. Arbitrary $f \in H^{s/2}(\Gamma)$ may be approximated in the given norm by f_n of the above type. According to (i) and the equality

$$\mathcal{E}_\mu^s(f_n - f_m, f_n - f_m) = \mathcal{E}^{s+n-d}(\tilde{f}_n - \tilde{f}_m, \tilde{f}_n - \tilde{f}_m)$$

the derivatives $D^{\frac{s}{2}+\frac{n-d}{2}}\tilde{f}_n$ of the Riesz extensions $\tilde{f}_n$ of f_n possess a limit h in $L_2(\mathbb{R}^n)$. We set $\tilde{f} := I^{\frac{s}{2}+\frac{n-d}{2}}h$ and obtain

$$\mathcal{E}_\mu^s(f,f) = \mathcal{E}^{s+n-d}(\tilde{f},\tilde{f}) \ ,$$

and $\tilde{f}$ is the Riesz extension of f. The equality for $f, g \in H^{s/2}(\Gamma)$ is a consequence. □

5 Generated semigroups and fractal pseudodifferential equations

By the general theory (see [3], Section 1.3) any closed quadratic form $\mathcal{E}$ in a real Hilbert space H generates a **strongly continuous** semigroup $(T_t)_{t\geq 0}$ (of contractive symmetric operators) on H. (Recall that this means the following: $\mathrm{dom}(T_t) = H$, $\|T_t\| \leq 1$, T_t is symmetric, $\lim\limits_{t\searrow 0} \|T_t f - f\| = 0$ for $f \in H$ and $\lim\limits_{t\searrow 0} t^{-1}(T_t f - f) = Af$, where $\mathcal{E}(f,f) = \langle f, -Af\rangle_H$ on the domain of the operator A which is called **infinitesimal generator** of the semigroup.) In our case $H = L_2(\mu)$, $\mathcal{E} = \mathcal{E}_\mu^s$ with

$$\mathcal{E}_\mu^s(f,g) = \left\langle \sqrt{D_\mu^s}\, f, \sqrt{D_\mu^s}\, g \right\rangle_{L_2(\mu)}$$

according to Theorem 4.1. This implies the following for $A = -D_\mu^s$. (Recall that $0 < s < d \leq n$.)

5.1 Corollary. *The operator $-D_\mu^s$ corresponding to the quadratic form $\mathcal{E}_\mu^s$ is the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on the real Hilbert space $L_2(\mu)$.*

This semigroup $(T_t)_{t\geq 0}$ is also denoted by

$$(e^{-tD_\mu^s})_{t\geq 0} \ .$$

As an application we consider the following **pseudodifferential equation** on the fractal support Γ of μ. For $s + n - d = 2$ it may be interpreted as the trace on Γ of the euclidean **heat equation**. Denote $R_+ := [0, \infty)$.

5.2 Theorem. *The equation*

$$\frac{\partial u}{\partial t}(t,\cdot) = -D_\mu^s u(t,\cdot) \ ,$$

$t \in \mathbb{R}_+$, with initial condition $u(0,\cdot) = f$ for some $f \in L_2(\mu)$ has the unique solution

$$u(t,\cdot) = e^{-tD_\mu^s} f$$

within the class $C(R_+, L_2(\mu))$.

5.3 Remark. The partial derivative $\frac{\partial u}{\partial t}(t,\cdot)$ on the left side of the equation is defined as the $L_2(\mu)$-limit

$$\lim_{s\to 0} \frac{1}{s}(u(t+s,\cdot) - u(t,\cdot)) \, .$$

Proof. of Theorem 5.2. It follows from Corollary 5.1 and the semigroup property that $e^{-tD_\mu^s} f$ is a solution with the above property.
Uniqueness in our case also follows from the general theory of strongly continuous semigroups. The following alternative procedure, which is similar as in the euclidean case, may be applied to more general linear pseudodifferential equations than that under consideration. Let $e_1(x),\ e_2(x), \ldots$ be a complete orthonormal system of eigenfunctions of the Riesz potential I_μ^s in $L_2(\mu)$. Then e_i is also an eigenfunction of the inverse operator D_μ^s with eigenvalue, say, λ_i . If u is a solution of the above problem then it may be decomposed by

$$u(t,x) = \sum_{i=1}^{\infty} u_i(t)\ e_i(x)$$

with continuous $u_i(t) = \langle u(t,\cdot), e_i\rangle_{L_2(\mu)}$. Moreover, existence of $\frac{\partial u}{\partial t}(t,\cdot)$ implies that of the derivatives $u_i'(t)$ and

$$\frac{\partial u}{\partial t}(t,\cdot) = \sum_{i=1}^{\infty} u_i'(t)\ e_i(x)\ .$$

On the other hand, $u(t,\cdot)$ belongs to the space of Riesz potentials ($D_\mu^s u(t,\cdot)$ is determined) which yields

$$-D_\mu^s u(t,\cdot) = \sum_{i=1}^{\infty} u_i(t)\ (-D_\mu^s e_i) = \sum_{i=1}^{\infty} u_i(t)\ (-\lambda_i)\ e_i\ .$$

Comparing the coefficients of e_i in the pseudodifferential equation we obtain

$$u_i'(t) = -\lambda_i\ u_i(t)\ ,\quad t \in \mathbb{R}_+\ ,$$

and from this

$$u_i(t) = u_i(0)\ e^{-\lambda_i t}\ .$$

Consequently,

$$u(t,x) = \sum_{i=1}^{\infty} u_i(0)\ e^{-\lambda_i t}\ e_i(x)\ .$$

The initial condition leads to $u_i(0) = \langle f, e_i\rangle_{L_2(\mu)}$. Thus,

$$u(t,x) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\ \langle f, e_i\rangle_{L_2(\mu)}\ e_i(x)$$

which agrees with the spectral representation of the above solution. □

5.4 Remark. Similarly as in the last proof instead of **fractional heat equations** the corresponding **fractional wave equations** may be solved. This will be shown elsewhere.

6 Dirichlet forms

In Theorem 4.1 we have shown that $\mathcal{E}_\mu^s$ is a regular closed quadratic form on the Hilbert space $L_2(\mu)$ which is again assumed to be real. Such a form is called a regular **Dirichlet form** if it additionally possesses the **Markov property**. We will work with the first of the following definitions (cf. [3]):

6.1 Definition. Let $(X, \mathfrak{X}, m)$ be a σ-finite measure space and $\mathcal{E}$ be a closed symmetric form on $L_2(X\ m)$ with domain $\mathcal{D}(\mathcal{E})$. Then $\mathcal{E}$ is said to be **Markovian** if one of the following equivalent conditions holds:

(i) If $f \in \mathcal{D}(\mathcal{E})$ and $\tilde{f} := (0 \vee f) \wedge 1$ then $\tilde{f} \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(\tilde{f}, \tilde{f}) \leq \mathcal{E}(f, f)$.

(ii) If $f \in \mathcal{D}(\mathcal{E})$ and g is a normal contraction of f then $g \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(g, g) \leq \mathcal{E}(f, f)$.

Here a function g is called a **normal contraction** of f if $|g(x)| \leq |f(x)|$ and $|g(x) - g(y)| \leq |f(x) - f(y)|,\ x, y \in X$. Note that the **truncation operator**

$$T(f) := (0 \vee f) \wedge 1$$

is a normal contraction. Moreover, it is a so called **composition operator**, i.e., $T(f) = c(f)$ for a continuous function $c : \mathbb{R} \to \mathbb{R}$ (which is given here by $c(r) := (0 \vee r) \wedge 1$).

It is well-known that the regular and closed quadratic form $\mathcal{E}^\sigma$ on $L_2(\mathbb{R}^n)$ with domain $H^{\sigma/2}(\mathbb{R}^n)$ considered above has the Markov property if and only if $0 < \sigma \leq 2$. This is shown by means of a close interplay with associated Markov semigroups and Fourier analysis (see, e.g. Jacob [5], Examples 4.5.23 and 5.7.28 and [3], Example 1.5.2).
Using the method of traces we will derive now the Markov property for the corresponding forms $\mathcal{E}_\mu^s$ if $s + n - d \leq 2$. (This further restriction is due to the techniques in the present paper .) We need the following auxiliary result. Let tr_Γ be the trace operator described in Section 3.)

6.2 Lemma. *For any $f \in H^\alpha(\mathbb{R}^n)$ and any d-set Γ as above with $0 \leq \frac{n-d}{2} < \alpha \leq 1$ and the truncation operator T we have*

$$tr_\Gamma T(f) = T(tr_\Gamma f)\ .$$

Proof. For $0 < \alpha \leq 1$ the space $H^\alpha(\mathbb{R}^n)$ has an equivalent norm given by the sum of the L_2-norm and a seminorm determined by differences of the functions. Since the truncation operator is a normal contraction, it follows that it is bounded in $H^s(\mathbb{R}^n)$. Every bounded composition operator in $H^s(\mathbb{R}^n)$ is continuous. For $\alpha = 1$ this is proved in [8]. For the case $0 < \alpha < 1$ it is derived in [10], Theorem

3, p. 377, by interpolation methods for nonlinear operators. (Continuity of the truncation operator can also be shown directly using the representation

$$\begin{aligned} c(x) - c(y) &= \int_0^1 c'(\lambda x + (1-\lambda)y)\ d\lambda\ (x-y) \\ &= \int_0^1 1_{[0,1]}(\lambda x + (1-\lambda)y)\ d\lambda\ (x-y) \end{aligned}$$

for the associated composition function c together with the above equivalent norm in $H^\alpha(\mathbb{R}^n)$.)
Let $\kappa_\varepsilon,\ \varepsilon \to 0$, be a usual family of smoothing kernels such that $f * \kappa_\varepsilon$ converges to f in $H^\alpha(\mathbb{R}^n)$. Then by continuity, $T(f * \kappa_\varepsilon)$ tends to $T(f)$ in the same sense. $H^\alpha(\mathbb{R}^n)$ is embedded into $B_{2,1}^{\frac{n-d}{2}}$ for $\frac{n-d}{2} < \alpha$. Therefore the definition of the trace operator tr_Γ implies the $L_2(\mu)$-convergences

$$\lim_{\varepsilon \to 0} f * \kappa_\varepsilon|_\Gamma = tr_\Gamma f$$

and

$$\lim_{\varepsilon \to 0} T(f * \kappa_\varepsilon)|_\Gamma = tr_\Gamma T(f)$$

for any d-measure μ with support Γ.
Since T is the truncation operator the first convergence yields also

$$\lim_{\varepsilon \to 0} T(f * \kappa_\varepsilon)|_\Gamma = T(tr_\Gamma f)$$

in $L_2(\mu)$. Hence, $T(tr_\Gamma f) = tr_\Gamma T(f)$. □

From this and Theorem 4.1 we conclude the following.

6.3 Theorem. *Let $0 < s < d \le n$ and $s + n - d \le 2$. Then the quadratic form*

$$\mathcal{E}_\mu^s(f,g) = \left\langle \sqrt{D_\mu^s}\, f, \sqrt{D_\mu^s}\, g \right\rangle_{L_2(\mu)}$$

on $L_2(\mu)$ is a regular Dirichlet form with domain $H^{s/2}(\Gamma)$.

Proof. According to Theorem 4.1 it remains to show the Markov property $\mathcal{E}_\mu^s$. For, we will prove that $f \in H^{s/2}(\Gamma)$ implies

$$T(f) \in H^{s/2}(\Gamma) \ \text{ and } \ \mathcal{E}_\mu^s(T(f), T(f)) \le \mathcal{E}_\mu^s(f,f)$$

for the truncation operator T.
The first assertion follows from Lemma 6.2.

Let $\tilde{f}$ be the Riesz extension of f as in Theorem 4.1 (iii). Below we will show that

$$\mathcal{E}^s_\mu(T(f),T(f)) \leq \mathcal{E}^{s+n-d}(T(\tilde{f}),T(\tilde{f})) \; . \tag{6.1}$$

Recall that $\mathcal{E}^{s+n-d}$ is a Dirichlet form in $L_2(\mathbb{R}^n)$ with domain $H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n)$, but it is defined for all functions g from the spaces of Riesz potentials $I^{\frac{s}{2}+\frac{n-d}{2}}(L_2(\mathbb{R}^2))$. Moreover, by Fourier analytical methods and the Levy-Chintchin representation one obtains for such g and $s+n-d<2$

$$\begin{aligned} \mathcal{E}^{s+n-d}(g,g) &= \langle D^{\frac{s}{2}+\frac{n-d}{2}} g, D^{\frac{s}{2}+\frac{n-d}{2}} g\rangle_{L_2(\mathbb{R}^n)} \\ &= \text{const} \iint \frac{(g(x)-g(y))^2}{|x-y|^{s+n-d+n}}\, dx\, dy \end{aligned} \tag{6.2}$$

(cf. [3], Example 1.4.1, [5], 5.7.28). Since

$$(T(g)(x)-T(g)(y))^2 \leq (g(x)-g(y))^2$$

this implies the Markov property of $\mathcal{E}^{s+n-d}$ on the larger space $I^{\frac{s}{2}+\frac{n-d}{2}}(L_2((\mathbb{R}^n))$. In particular,

$$\mathcal{E}^{s+n-d}(T(\tilde{f}),T(\tilde{f})) \leq \mathcal{E}^{s+n-d}(\tilde{f},\tilde{f}) \; .$$

The right side equals $\mathcal{E}^s_\mu(f,f)$ in view of Theorem 4.1 (iii). Hence,

$$\mathcal{E}^s_\mu(T(f),T(f)) \leq \mathcal{E}^s_\mu(f,f) \; .$$

It remains to prove the inequality (6.1). The trace operator $tr_\Gamma : H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n) \to H^{s/2}(\Gamma)$ may be extended to $I^{\frac{s}{2}+\frac{n-d}{2}}(L_2(\mathbb{R}^n))$ as follows: By construction we have for $g \in H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n)$

$$\mathcal{E}^s_\mu(tr_\Gamma g, tr_\Gamma g) = ||tr_\Gamma g| \; H^{\frac{s}{2}}_R(\Gamma)\| \leq \|g| \; H^{\frac{s}{2}+\frac{n-d}{2}}_R(\mathbb{R}^n)\| \; .$$

Taking the limit as $R\to\infty$ on the right-hand side, which agrees with

$$\int \lim_{R\to\infty} (D_R^{\frac{s}{2}+\frac{n-d}{2}} g(x))^2 dx = \int (D^{\frac{s}{2}+\frac{n-d}{2}} g(x))^2\, dx = \mathcal{E}^{s+n-d}(g,g) \; ,$$

we obtain

$$\mathcal{E}^s_\mu(tr_\Gamma g, tr_\Gamma g) \leq \mathcal{E}^{s+n-d}(g,g) \; .$$

Since $H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n)$ is dense in the space of Riesz potentials $I^{\frac{s}{2}+\frac{n-d}{2}}(L_2(\mathbb{R}^n))$, the continuous extension

$$tr_\Gamma : I^{\frac{s}{2}+\frac{n-d}{2}}(L_2(\mathbb{R}^n)) \to H^{s/2}(\Gamma)$$

is determined.
Furthermore, as in the proof of Lemma 6.2 we get from (6.2) for any $g \in I^{\frac{s}{2}+\frac{n-d}{2}}(L_2(\mathbb{R}^n))$,

$$T(tr_\Gamma g) = tr_\Gamma T(g) \ .$$

This yields

$$\mathcal{E}^s_\mu(T(tr_\Gamma g), T(tr_\Gamma g)) \leq \mathcal{E}^{s+n-d}(T(g), T(g))$$

and, in particular, (6.1).
The case $s + n - d = 2$ may be treated analogously when replacing (6.2) by

$$\mathcal{E}^2(g, g) = \int |\mathrm{grad}\, g(x)|^2 \, dx \ .$$

□

6.4 Remark.

(i) The above Dirichlet forms generate fractal variants of symmetric stable Levy processes. Other stable-like jump processes on Γ (and in more general metric spaces) are discussed in Kumagai [7] and Stós [13].

(ii) In Hu and Lau [4] it is shown that the Green's kernel of $I^{d_w}_\mu$ for the "walk dimension" d_w of certain generalized Sierpinski carpets Γ is comparable to that of the potential operator of the Brownian motion constructed by Barlow and Bass [1].(See Mosco [9] for the corresponding Laplace operator w.r.t. an appropriate inner quasimetric.)

(iii) A survey on [17] and the present topic may be found in [18].

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