Math. Nachr. **278,** No. 9, 1097 – 1106 (2005) / **DOI** 10.1002/mana.200310295

Stochastic differential equations with fractal noise

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Received 8 October 2002, revised 23 July 2003, accepted 24 July 2003
Published online 8 June 2005

Key words Forward integral, quadratic variation process, anticipating stochastic differential equation, Besov space

MSC (2000) 60H10, 60H05, 60H15, 34F05

Stochastic differential equations in $\mathbb{R}^n$ with random coefficients are considered where one continuous driving process admits a generalized quadratic variation process. The latter and the other driving processes are assumed to possess sample paths in the fractional Sobolev space W_2^β for some $\beta > 1/2$. The stochastic integrals are determined as anticipating forward integrals. A pathwise solution procedure is developed which combines the stochastic Itô calculus with fractional calculus via norm estimates of associated integral operators in W_2^α for $0 < \alpha < 1$. Linear equations are considered as a special case.

This approach leads to fast computer algorithms basing on Picard's iteration method.

1 Introduction

For applications in mathematical finance (see, e.g. [10]) and other fields the following stochastic differential equation in $\mathbb{R}^n$ is relevant:

$$\begin{aligned} dX(t) &= a_0\big(X(t),t\big)\,dW(t) + \sum_{j=1}^{m} a_j\big(X(t),t\big)\,dB^{H_j}(t) + b\big(X(t),t\big)\,dt\,, \\ X(t_0) &= X_0\,. \end{aligned}$$

Here W is the one-dimensional Wiener process and the B^{H_j} denote fractional Brownian motions with Hurst exponents H_j which may vary in time. They may be introduced by means of the representation

$$B^{H_j}(t) := \int \frac{e^{itu}-1}{|u|^{H_j(t)+1/2}}\,dW^j(u)$$

for Wiener processes $W^1,\dots,W^m$. (For the constant H_j we obtain classical fractional Brownian motion.) We will consider only the case $H_j(t) \geq H > \frac{1}{2}$ and assume that all $H_j(t)$ are Hölder continuous functions. In this case the sample paths of the B^{H_j} have almost surely nice fractional smoothness properties: They are Hölder continuous of all orders less than H and they possess finite p variations, $p > \frac{1}{H}$, and fractional derivatives of all orders less than H. Moreover, they are elements of the Sobolev–Slobodeckij (or Besov) spaces W_2^{H-} which are most appropriate to our approach. This guarantees that under smoothness assumptions on the random vector fields a_i the stochastic integrals in the second summand of the above equation for a suitable notion of solution may be understood in the sense of a.s. convergence of the Riemann–Stieltjes sums. The stochastic integral for the first summand, i.e. the Brownian motion component, may be determined in the sense of uniform convergence in probability of the Riemann–Stieltjes sums. (We do not assume any adaptedness on the random vector fields $a_0, a_1, \dots, a_m, b$.)

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In particular, the corresponding linear equation may be used for modelling stock price developments and option pricing in mathematical finance. The fractional Brownian motion components lead to long range dependence and the Wiener process guarantees strong no-arbitrage (see [15]).

In general, the above equation cannot be treated by means of semimartingale theory (for adapted coefficients). Because of the Brownian motion component it also does not fit into the models using finite p-variations, Hölder conditions, or fractional derivatives, where the solution depends continuously on the driving processes. These suppose integrands and integrators of summed order of fractional smoothness greater than 1. In recent more abstract papers the case of lower order Hurst exponents has been considered for different types of stochastic integrals, but the solution procedures are complicated . A numerically more accessible approach to the pathwise solution consists in the following. (For simplicity it is demonstrated here on the time homogeneous case, the general case will be treated below.) As in the Doss–Sussman approach for $m = 0$ we seek the solution pathwise in the form

$$X(t) \ = \ h(Y(t), W(t))$$

for some smooth function h satisfying

$$\frac{\partial h}{\partial y}(y, z) \ = \ a_0(h(y, z))$$

and an unknown vector process $Y(t)$. Then we apply the Itô formula which will be adopted as a general calculation rule in order to determine an auxiliary stochastic differential equation where $dW(t)$ is eliminated:

$$dY(t) \ = \ \sum_{i=1}^{m} \tilde{a}_i(Y(t), W(t))\, dB^{H_i}(t) + \tilde{b}(Y(t), W(t))\, dt$$

with new random vector fields $\tilde{a}_1, \ldots \tilde{a}_m, \tilde{b}$. Here $W(t)$ may be considered as a parameter function (of less order of smoothness than that of the integrators). Such type of equations have been treated in our paper [13] in a more special situation. Extending this approach to mappings depending on parameter functions as above we get pathwise a unique local solution $Y(t)$ with coordinates in the Besov space W_2^{H-}. Substituting this Y (depending on the choice of h) in the above formula for $X(t)$ we indeed obtain a solution. We will show that the solution is unique in the class of all processes with generalized quadratic variations satisfying the Itô calculation rule. (For different h we obtain different representations of the same process.) Note that the (random) function h and the auxiliary process Y may be determined by Picard's iteration method.

The essential part of our approach is the above auxiliary SDE for the process Y. It is solvable for more general driving processes and parameter processes than $B^{H_1}, \ldots, B^{H_m}$ and W, resp. Moreover, the method of showing convergence in probability of the Riemann–Stieltjes sums for the first integral in the above equation by means of the Taylor expansion is well-known from the literature and goes back to Föllmer [5]. In contrast to other papers we derive this convergence from that of the remaining integrals. The latter can be shown by at least three different methods using pathwise fractional smoothness properties of integrands and integrators of summed order greater than 1 (cf. Section 2.1). The Wiener process W will be replaced by an arbitrary continuous process Z with generalized quadratic variation $[Z] \in W_2^{H-}$ if the first integral is understood in a similar sense. Instead of fractional Brownian motions B^H we will choose arbitrary processes with sample paths in W_2^{H-}, where $H > 1/2$.

2 Stochastic integrals and processes with generalized quadratic variation

2.1 Convergence of Riemann–Stieltjes sums and related function spaces

The following deterministic models may be applied to the sample paths of stochastic processes. For real-valued functions f and g on an interval $[0, T]$ the extended Lebesgue–Stieltjes integrals

$$\int_0^t f\, dg\,, \quad t \in (0, t]\,,$$

have been determined in the following three situations and agree with each other on the common spaces of definition:

(i) L. C. Young [11] proved (non-absolute uniform in t) convergence of the Riemann–Stieltjes sums in the case where f and g have finite **p**- and **q**-variations $\mathrm{Var}_p f$ and $\mathrm{Var}_q g$ with $\frac{1}{p}+\frac{1}{q} > 1$ and do not possess common points of discontinuity. Moreover,

$$\left| \int_0^t f\, dg \right| \leq \mathrm{Var}_p f \,\mathrm{Var}_q g + |f(0)\,(g(t)-g(0))| \,.$$

A stochastic versions was considered in Bertoin [1]. (An introduction to this approach and more recent developments may be found in Dudley and Norvaiša [3].)

(ii) For smooth functions f and g Feyel and de La Pradelle [4] derived the estimate

$$\left| \int_0^t f\, dg \right| := \left| \int_0^t f\, g'\, ds \right| \leq \mathrm{const}\, \|f\|_\alpha \, \|g\|_\beta \, t^{1+\varepsilon} + |f(0)\,(g(t)-g(0))|$$

where $\|\cdot\|_\alpha$ denotes the Hölder norm of order α and $0 < \varepsilon < \alpha+\beta-1$. Therefore the integral extends continuously to Hölder functions f and g of summed order greater than 1. For such functions they proved uniform (in t) convergence of the Riemann–Stieltjes sums to that integral.

(iii) In our paper [12] we introduced

$$\int_s^t f\, dg := (-1)^\alpha \int_s^t D^\alpha_{s+} f_{s+}(u)\, D^{1-\alpha}_{t-} g_{t-}(u)\, du + f(s+)(g(t-)-g(s+))$$

where $f_{s+}(u) := f(u) - f(s+)$, $g_{t-}(u) := g(u) - g(t-)$ and $D^\beta_{0+}\varphi$ $\left(D^\beta_{t-}\varphi\right)$ denote the *left-sided* (resp. *right-sided*) *fractional derivative of order* $\beta \in (0,1)$ of a function φ on the interval (s,t) in the sense of [9]. The derivatives $D^\alpha_{s+} f_{s+}$ and $D^{1-\alpha}_{t-} g_{t-}$ are assumed to be elements of L_p and L_q, resp., where $1/p + 1/q = 1$. (The definition does not depend on the choice of α and the corresponding properties of an integral are proved.)

For the special case of Hölder continuous functions of summed order greater than 1 (cf. (ii)) uniform convergence of the Riemann–Stieltjes sums to our integral is shown with intrinsic methods of fractional calculus. Thus, in this case the integrals in (i), (ii) and (iii) agree. It turns our that for the purposes of the present paper the more general Sobolev–Slobodeckij spaces W^α_2 (which coincide up to norm equivalence with the Besov spaces $B^\alpha_{2,2}$) are most appropriate. The norms in these Banach spaces are given by

$$\|f\|_{W^\alpha_2} := \left(\int_0^T f(t)^2\, dt \right)^{1/2} + \left(\int_0^T \int_0^T \frac{(f(s)-f(t))^2}{|s-t|^{2\alpha+1}}\, ds\, dt \right)^{1/2}$$

where $0 < \alpha < 1$.

In order to prove contraction properties of a related integral operator it is appropriate to replace the L_2-norm of the function itself by the L_∞-norm. (For $\alpha > 1/2$ this does not change the space W^α_2 since these functions are continuous.)

Denote

$$\|f\|_{\widetilde{W}^\alpha_2} := \left(\int_0^T \int_0^T \frac{(f(s)-f(t))^2}{|s-t|^{2\alpha+1}}\, ds\, dt \right)^{1/2},$$

$$\|f\|_{W^\alpha_{2,\infty}} := \|f\|_{L_\infty} + \|f\|_{\widetilde{W}^\alpha_2} \,.$$

When restricting to subintervals (s,t) we write $W^\alpha_2(s,t)$, etc. Further,

$$W^{\beta-}_2 := \bigcap_{\alpha<\beta} W^\alpha_2 \,, \quad \text{etc.}$$

The *Liouville spaces* $I^\alpha_{s+}(L_2)$ (resp. $I^\alpha_{t-}(L_2)$) (used in the definition of the integral) are given by the norms

$$\|f\|_{I^\alpha_{s+\,(t-)}(L_2)} := \|f\|_{L_2} + \|D^\alpha_{s+\,(t-)} f\|_{L_2} \sim \|D^\alpha_{s+\,(t-)} f\|_{L_2} \,.$$

Relationships between these norms may be found, e.g., in [4] and [13].

In [13] it is also shown that the integral in 2.1 (iii) may be represented in the form

$$\int_0^t f\,dg \;=\; \lim_{\varepsilon\to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t f(s)\, \frac{g_{t-}(s+u)-g_{t-}(s)}{u}\, ds\, du \tag{2.1}$$

where convergence holds uniformly in $t \in (0,T]$. The limit on the right-hand side exists under more general assumptions on f and g and may be considered as an extension of our integral. Note that the kernel $\varepsilon u^{\varepsilon-1}$ acts as the δ-function as $\varepsilon \to 0$. Therefore the existence of

$$\lim_{u\searrow 0} \int_0^t f(s)\, \frac{g_{t-}(s+u)-g_{t-}(s)}{u}\, ds$$

implies that of the above limit. In the context of uniform convergence in probability the latter corresponds to a slight extension of the so-called *forward integral* of Russo and Vallois [6] for stochastic processes. More details and relationships to other integrals may be found in [14].

2.2 Stochastic forward integrals, quadratic variations and Itô formulae

Suppose now that Y is a stochastic càglàd (left continuous with right limits) process and Z a stochastic càdlàg (right continuous with left limits) process on $[0,T]$. Then we define (cf. [13])

$$\left(\int_0^{t-} Y\,dZ \;=\right) \int_0^t Y\,dZ \;:=\; \lim_{\varepsilon\to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t Y(s)\, \frac{Z_{t-}(s+u)-Z_{t-}(s)}{u}\, ds\, du \tag{2.2}$$

whenever the right-hand side is determined, where $\lim$ stands for uniform convergence in probability and $\int_0^1$ for $\lim_{\delta\searrow 0} \int_\delta^1$ with probability 1. Then

$$X(t) \;:=\; \int_0^{t+} Y\,dZ$$

is càdlàg and $X(t)-X(t-) = Y(t)(Z(t)-Z(t-))$. Moreover, continuity of Z implies that of X.

Remark 2.1 Russo and Vallois use the same stochastic forward integral, but without averaging in the limit procedure. If Z is a semimartingale and Y an adapted càglàd process then the integral (2.1) agrees with the usual Itô integral $\int_{0+}^{t-} Y\,dZ$ (see [14]).

The quadratic variation process (or bracket) $[Z](t)$ is a well-known notion in semimartingale theory. Having in mind the stochastic integral (2.2) we now define the *generalized quadratic variation process* (*bracket*)

$$[Z](t) \;:=\; \lim_{\varepsilon\to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_0^t \frac{1}{u}\, (Z_{t-}(s+u)-Z_{t-}(s))^2\, ds\, du + (Z(t)-Z(t-))^2 \tag{2.3}$$

for any càdlàg process Z such that convergence holds uniformly (in $t \in (0,T]$) in probability. The *generalized covariation process* $[Y,Z]$ of two such processes is introduced analogously. For the special case of semimartingales these notions agree with the classical ones. Many properties of our extensions may be proved similarly as in Russo–Vallois [7] where the case of non-averaged limits is treated. (For this and the following relationships see [13].) Let now Z be a continuous process with generalized bracket $[Z]$. Then we get for any random C^1-function $F(z,t)$ on $\mathbb{R}\times[0,T]$ with continuous $\frac{\partial^2 F}{\partial z^2}$ and for $0 \le s < t \le T$ the *simple Itô formula*

$$\begin{aligned} &F(Z(t),t) - F(Z(s),s) \\ &= \int_s^t \frac{\partial F}{\partial x}(Z(u),u)\, dZ(u) + \int_s^t \frac{\partial F}{\partial t}(Z(u),u)\, du + \frac{1}{2}\int_s^t \frac{\partial^2 F}{\partial x^2}(Z(s),s)\, d[Z](s) \end{aligned} \tag{2.4}$$

and the stochastic integral is determined in the sense of (2.2). Further, by definition the process X with

$$X(t) \;=\; \int_0^t A\,dZ$$

for some càglàd process A satisfies the general Itô formula if

$$\begin{aligned} &F(X(t),t) - F(X(s),s) \\ &= \int_s^t \frac{\partial F}{\partial x}(X(u),u)A(u)\,dZ(u) + \int_s^t \frac{\partial F}{\partial t}(X(u),u)\,du \\ &\quad + \frac{1}{2}\int_s^t \frac{\partial^2 F}{\partial x^2}(X(u),u)A(u)^2\,d[Z](u)\,. \end{aligned} \tag{2.5}$$

Higher-dimensional extensions are straightforward. For our purposes we need the following special version. First note as an important consequence of Definition (2.3) that any continuous process Z admitting a generalized bracket $[Z]$ belongs to the space $W_2^{1/2-}$ with probability 1.

$F(y,z)$ is now an $\mathbb{R}^n$-valued random C^1-function on $\mathbb{R}^m \times \mathbb{R}$ with continuous $\frac{\partial^2 F}{\partial z^2}$ and $Y = (Y^1,\dots,Y^m)$ is a random process with coordinate sample paths in $W_{2,\infty}^{\beta}$ for some $\beta > \frac{1}{2}$. Then we obtain

$$\begin{aligned} &F(Y(t),Z(t)) - F(Y(s),Z(s)) \\ &= \sum_{i=1}^m \int_s^t \frac{\partial F}{\partial y^i}(Y(u),Z(u))\,dY^i(u) + \int_s^t \frac{\partial F}{\partial z}(Y(u),Z(u))\,dZ(u) \\ &\quad + \frac{1}{2}\int_s^t \frac{\partial^2 F}{\partial z^2}(Y(u),Z(u))\,d[Z](u)\,. \end{aligned} \tag{2.6}$$

Here the first m integrals are determined in the sense of Section 2.1 (iii) or, equivalently, by a.s. convergence of the Riemann–Stieltjes sums. The integral w.r.t. Z is given by the stochastic forward integral (2.2). (The assumption on the Y_i implies that these are processes of vanishing quadratic (co)variations. Similarly, the covariations of Y_i and Z are zero.) Note that we may choose, in particular, $Y_m(t) = t$ in order to apply the formula to time dependent SDE.

Let now $Z^0 := Z$ be as before and suppose that $Z^1,\dots,Z^m$ are processes with sample paths in W_2^{β} for some $\beta > \frac{1}{2}$. (Recall that such processes are continuous.)

Definition 2.2 The vector process X with integral representation

$$X(t) \;=\; X(0) + \sum_{i=0}^m \int_0^t A_i\,dZ^i$$

admits the (*random*) *Itô calculus* if

$$\big[X^i,X^k\big](t) \;=\; \int_0^t A_0^j\,A_0^k\,d\big[Z^0\big]$$

exist and for any (random) vector function F (under consideration) of class C^2 the general Itô formula holds true:

$$\begin{aligned} &F(X(t)) - F(X(s)) \\ &= \sum_{j=1}^n \sum_{i=0}^m \int_s^t \frac{\partial F}{\partial x^j}(X(u))\,A_i^j(u)\,dZ^i + \frac{1}{2}\sum_{j,k=1}^n \int_s^t \frac{\partial^2 F}{\partial x^j\,\partial x^k}(X(u))\,A_0^j(u)\,A_0^k(u)\,d\big[Z^0\big](u)\,. \end{aligned} \tag{2.7}$$

(The integrals w.r.t. to Z^i are defined in the sense of (2.2).) Again, the time dependent case is included taking $Z^{m+1}(t) = t,\ X^{n+1}(t) = t,\ A_i^{n+1} \equiv 0,\ i \le m,\ A_{m+1}^n \equiv 1$. Then the smoothness of F w.r.t. to the time argument may be relaxed as in (2.6) for y.

3 An integral operator and its contraction property

We now turn back to the deterministic case and consider integrator functions $g \in W_2^{H-}$ always assuming that $\boldsymbol{H > \frac{1}{2}}$. Later on g will be replaced by fractional Brownian motion B^H or general stochastic processes with sample paths in W_2^{H-}. We also consider a parameter function $\varphi \in W_{2,\infty}^{1/2-}$. In the stochastic version φ corresponds to the Wiener process W or, more generally to an arbitrary continuous process Z with generalized quadratic variation $[Z]$.

For fixed $t_0 \in [0,T)$ we consider the integral operator

$$Af := x_0 + \int_{t_0}^{(\cdot)} a(f,\varphi)\, dg$$

for some smooth function a. For $0 < \gamma \le 1$, $t_0 < t < T$ and $C > 0$ let $W_{2,\infty}^{\gamma}(t_0,t;\, x_0, C)$ be the set of functions on (t_0,t) such that $f(t_0+) = x_0$ and $\|f_{t_0+}\|_{W_{2,\infty}^{\gamma}(t_0,t)} \le C$.

The following result provides a local contraction principle for the operator A:

Theorem 3.1 *Let $x_0, y_0 \in \mathbb{R}$, $a \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\frac{\partial a}{\partial x}(x,y)$, $\frac{\partial a}{\partial y}(x,y)$ be locally Lipschitz in x and $g \in W_2^{H-}$ for some $1/2 < H \le 1$. Then for any $1/2 < \beta < H$ and positive constants c, C and K there is some $t \in (t_0,T)$ such that for any $\varphi \in W_{2,\infty}^{1/2-}(t_0,t;\, y_0, K)$ the integral operator*

$$Af = x_0 + \int_{t_0}^{(\cdot)} a(f,\varphi)\, dg$$

maps $W_{2,\infty}^{\beta}(t_0,t;\, x_0,C)$ into itself and we have

$$\|Af - Ah\|_{W_{2,\infty}^{\beta}(t_0,t)} \le c\, \|f-h\|_{W_{2,\infty}^{\beta}(t_0,t)}$$

for all $f,h \in W_{2,\infty}^{\beta}(t_0,t;\, x_0,C)$.

Proof. In Theorem 2.3 of [13] we have shown this under the stronger assumption that $\varphi \in W_{2,\infty}^{\beta}(t_0,t;\, y_0,K)$ which was actually not needed. The arguments remain valid when using Theorem 2.2 (i) for $\alpha := 1-H+\frac{H-\beta}{2}$ instead of Theorem 2.2 (ii) from [13] and regarding in the norm estimates that $\alpha < \alpha+\beta-\frac{1}{2} = \frac{1}{2} - \frac{H-\beta}{2} < \frac{1}{2}$. □

Note that the above integrals are determined in the sense of Section 2.1. We now will replace the function a by a vector field and define the integrals coordinatewise. Then we get the following straightforward higher-dimensional extension w.r.t. to the usual norm in Cartesian products of normed spaces.

Theorem 3.2 (i) *The statement of Theorem* 3.1 *remains valid if $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^k$, $g^j \in W_2^{H-}$, $a_j \in C^1(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^n)$ with partial derivatives being locally Lipschitz in the first n arguments, $j = 1,\dots,l$, φ takes values in $\mathbb{R}^k$ and f and h in $\mathbb{R}^n$ with coordinate functions as before and the integral operator is given by*

$$Af := x_0 + \sum_{j=1}^{l} \int_{t_0}^{(\cdot)} a_j(f,\varphi)\, dg^j .$$

In particular, we obtain the local contraction principle for this integral operator.

(ii) *If $g^j(t) = t$ for some j then the conditions on a_j may be relaxed to measurability, local boundedness and local Lipschitz continuity w.r.t. the first n arguments.*

Remark 3.3 The modifications in the corresponding proofs of [13] for the case (ii) are straightforward.

4 Auxiliary differential equations

Turning back to our primary aim we now consider on $\mathbb{R}^n \times \mathbb{R} \times [0,T]$ the $\mathbb{R}^n$-valued C^1-vector fields $\tilde{a}_j(y,z,t)$ such that all partial derivatives are locally Lipschitz in the first n variables y, $j = 1,\dots,l$, and the locally bounded measurable vector field $\tilde{b}(y,z,t)$ being Lipschitz in y. Let $z^0(t)$ be a real-valued parameter function from $W_{2,\infty}^{1/2-}$ and $z^1,\dots,z^l \in W_2^{H-}$, $H > \frac{1}{2}$, be driving functions for the differential equation

$$\begin{aligned} dy(t) &= \sum_{j=1}^{m} \tilde{a}_j\big(y(t), z^0(t), t\big)\, dz^j(t) + \tilde{b}\big(y(t), z^0(t), t\big)\, dt\,, \\ y(t_0) &= y_0\,, \quad \text{where} \quad t_0 \in (0,T)\,. \end{aligned} \tag{4.1}$$

This equation becomes exact via integration in the sense of Section 2.1 (The last summand may be included into the sum defining $z^{m+1}(t) := t$. Considering the time argument t in the vector fields as an additional parameter function $\varphi(t) = t$ we obtain an integral equation corresponding to Theorem 3.2.)

Theorem 4.1 *Under the above conditions for any $\frac{1}{2} < \beta < H$ there exists an interval $(t_1, t_2) \subset [0, T]$ containing t_0 with the following properties. Equation (4.1) has a solution y on (t_1, t_2) with coordinate functions in $W^{\beta}_{2,\infty}(t_1, t_2)$. It may be determined by means of Picard's iteration method which is contractive. A $W^{1/2-}_{2,\infty}$-solution is unique on the maximal interval of definition and belongs to W^{H-}_2.*

Proof. The first part is completely analogous to that of Theorem 6.1 in [13] using Theorem 3.2 above. The contraction principle provides a unique local $W^{\beta}_{2,\infty}$-solution. If we have any local $W^{1/2-}_{2,\infty}$ solution y then Theorem 1.2 (iii) in [13] applied to the integral representation of y yields $y \in W^{\beta}_{2,\infty}$ on the interval of definition for any $\beta < H$. $\big($Choose there $\alpha := 1 - H + \frac{H-\beta}{2}$.$\big)$ □

This theorem is an essential tool for solving the above stochastic differential equations.

5 Stochastic differential equations

In this section our main results will be presented. First we choose (random) vector fields $a_0, a_1, \ldots, a_m, b$ satisfying the conditions

(C1) $a_j \in C^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$, all partial derivatives are locally Lipschitz in $x \in \mathbb{R}^n$,

(C2) $b \in C(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$ is locally Lipschitz in $x \in \mathbb{R}^n$

(with probability 1 in the random case). As in Section 2.2, Z^0 is a continuous process with generalized bracket $[Z^0]$ and $Z^1, \ldots, Z^m$ are processes with sample paths in W^{H-}_2 for some $H > 1/2$. X_0 is an arbitrary initial (random) value.

Definition 5.1 A *local solution* $X = (X^1, \ldots, X^n)$ of the SDE

$$\begin{aligned} dX(t) &= \sum_{j=0}^{m} a_j(X(t), t)\, dZ^j(t) + b\,(X(t), t)\ dt\,, \\ X(t_0) &= X_0 \end{aligned} \tag{5.1}$$

is a process with generalized quadratic variation admitting the (random) Itô calculus with respect to its integral representation

$$X(t) \;=\; X_0 + \sum_{j=0}^{m} \int_{t_0}^{t} a_j(X(s), s)\, dZ^j(s) + \int_{t_0}^{t} b(X(s), s)\, ds$$

in some neighborhood of t_0.

(Cf. Section 2.2. The stochastic integrals are defined by (2.2). For $j = 1, \ldots, m$ this is equivalent to a.s. convergence of the Riemann–Stieltjes sums. Continuity of X is a consequence of the integral representation.)

In order to determine a pathwise local solution we first proceed similarly as in Section 7 of [13]:

Consider the auxiliary pathwise differential equation on $\mathbb{R}^n \times \mathbb{R} \times [0, T]$

$$\begin{aligned} \frac{\partial h}{\partial z}(y, z, t) &= a_0(h(y, z, t), t)\,, \\ h(Y_0, Z_0, t_0) &= X_0\,, \end{aligned} \tag{5.2}$$

where $Z_0 := Z^0(t_0)$ and Y_0 is an arbitrary random vector in $\mathbb{R}^n$. Picard's iteration method provides a (non-unique) local solution $h \in C^1$ in a neighborhood of (Y_0, Z_0, t_0) with partial derivatives being Lipschitz in y and

$$\det\left(\frac{\partial h}{\partial y}(y, z, t)\right) \neq 0\,.$$

Moreover,

$$\frac{\partial^2 h}{\partial z^2}(y,z,t) \;=\; \sum_{i=1}^{n} \frac{\partial a_0}{\partial x^i}(h(y,z,t),t)a_0^i(h(y,z,t),t)\,.$$

We will seek the solution X of (5.1) in the form

$$X(t) \;=\; h\big(Y(t), Z^0(t), t\big)$$

for some random W_2^{H-}-process Y to be determined (in dependence of the choice of h and $Y(t_0) = Y_0$). Applying the Itô formula (6) to the function h we obtain

$$\begin{aligned} dX(t) \;&=\; \frac{\partial h}{\partial z}\big(Y(t), Z^0(t), t\big)\, dZ^0(t) + \sum_{k=1}^{n} \frac{\partial h}{\partial y^k}\big(Y(t), Z^0(t), t\big)\, dY^k(t) \\ &\quad+\; \frac{\partial h}{\partial t}\big(Y(t), Z^0(t), t\big)\, dt + \frac{1}{2}\, \frac{\partial^2 h}{\partial z^2}\big(Y(t), Z^0(t), t\big)\, d\big[Z^0\big](t) \\ &=\; a_0(X(t),t)\, dZ^0(t) + \sum_{k=1}^{n} \frac{\partial h}{\partial y^k}\big(Y(t), Z^0(t), t\big)\, dY^k(t) + \frac{\partial h}{\partial t}\big(Y(t), Z^0(t), t\big)\, dt \\ &\quad+ \frac{1}{2}\sum_{i=1}^{n} \frac{\partial a_0}{\partial x^i}\big(h\big(Y(t), Z^0(t), t\big), t\big) a_0^i\big(h\big(Y(t), Z^0(t), t\big), t\big)\, d\big[Z^0\big](t)\,. \end{aligned}$$

Comparing this with (5.1) we are led to a second auxiliary SDE:

$$\begin{aligned} &\sum_{k=1}^{n} \frac{\partial h}{\partial y^k}\big(Y(t), Z^0(t), t\big)\, dY^k(t) \\ &\quad= \sum_{j=1}^{m} a_j\big(h\big(Y(t), Z^0(t), t\big), t\big)\, dZ^j(t) + \Big(b\big(h\big(Y(t), Z^0(t), t\big), t\big) - \frac{\partial h}{\partial t}\big(Y(t), Z^0(t), t\big)\Big)\, dt \\ &\qquad- \frac{1}{2}\sum_{i=1}^{n} \frac{\partial a_0}{\partial x^i}\big(h\big(Y(t), Z^0(t), t\big), t\big)\, a_0^i\big(h\big(Y(t), Z^0(t), t\big), t\big)\, d\big[Z^0\big](t)\,. \end{aligned}$$

In a neighborhood of t_0 it is equivalent to following matrix representation.

$$\begin{aligned} dY(t) \;&=\; \Big(\frac{\partial h}{\partial y}\big(Y(t), Z^0(t), t\big)\Big)^{-1} \\ &\quad\times \Bigg[\sum_{j=1}^{m} a_j\big(h\big(Y(t), Z^0(t), t\big), t\big)\, dZ^j(t) \\ &\qquad+ \Big(b\big(h\big(Y(t), Z^0(t), t\big), t\big) - \frac{\partial h}{\partial t}\big(Y(t), Z^0(t), t\big)\Big)\, dt \\ &\qquad- \frac{1}{2}\,\frac{\partial a_0}{\partial x}\big(h\big(Y(t), Z^0(t), t\big), t\big) a_0\big(h\big(Y(t), Z^0(t), t\big), t\big)\, d\big[Z^0\big](t)\Bigg], \\ Y(t_0) \;&=\; Y_0\,. \end{aligned} \tag{5.3}$$

In order to apply Theorem 4.1 we now additionally assume $[Z^0] \in W_2^{H-}$ with the same H as for $Z^1, \ldots, Z^m$ and conclude the existence of a pathwise local solution $Y \in W_2^{H-}$ of (5.3) which is unique in $W_{2,\infty}^{1/2-}$.

It turns out that this procedure provides the unique solution of our stochastic differential equation:

Theorem 5.2 *Suppose that the random vector fields satisfy* (C1) *and* (C2)*,* Z^0 *is a continuous process with generalized bracket* $[Z^0]$ *and* $[Z^0]$*,* $Z^1, \ldots, Z^m$ *are processes with sample paths in* W_2^{H-} *for some* $H > 1/2$.

Then any representation

$$X(t) \;=\; h\big(Y(t), Z^0(t), t\big)$$

with h satisfying (5.2) *and $Y \in W_2^{H-}$ locally determined by* (5.3) *provides a pathwise local solution of the SDE* (5.1).

If X is an arbitrary solution in the sense of Definition 5.1 *then it agrees with any of the above representations on the common interval of definition.*

P r o o f. Exploiting Theorem 4.1 we can use completely the same arguments as in the proof of Theorem 7.1.1 in [13]. For brevity we recall only the main ideas. First note that applying the Itô formula to the function h and using (5.2) and (5.3) one shows that $h\big(X(t), Z^0(t), t\big)$ is indeed a solution. In order to prove uniqueness take any solution $X(t)$ as required and compare it with one of the above $h\big(Y(t), Z^0(t), t\big)$: The mapping

$$(y, z, t) \;\longrightarrow\; (h(y, z, t), z, t)$$

is invertible in a neighborhood of (Y_0, Z_0, t_0) with differentiable inverse mapping $(x, z, t) \to (u(x, z, t), z, t)$, i.e.,

$$u(h(y, z, t), z, t) \;=\; y\,.$$

Then the partial derivatives of u and h are in certain relationships. These and the Itô formula (2.7) applied to the random function u and the process $\big(X(t), Z^0(t)\big)$ given in integral representation

$$\begin{aligned} X(t) \;&=\; X_0 + \sum_{j=0}^{m} \int_{t_0}^{t} a_j(X(s), s)\, dZ^j(s) + \int_{t_0}^{t} b(X(s), s)\, ds\,, \\ Z^0(t) \;&=\; Z_0 + \int_{t_0}^{t} dZ^0(s) \end{aligned}$$

lead to an integral representation of $u\big(X(t), Z^0(t), t\big)$. Substituting $\frac{\partial h}{\partial z}(y, z, t) = a(h(y, z, t), t)$ one concludes that the process

$$\widetilde{Y}(t) \;:=\; u\big(X(t), Z^0(t), t\big)$$

has sample paths in $W_{2,\infty}^{1/2-}$ and satisfies the integral equation (5.3). Since the solution of (5.3) is unique in $W_{2,\infty}^{1/2-}$ we infer $\widetilde{Y}(t) = Y(t)$, hence $u\big(X(t), Z^0(t), t\big) = Y(t)$ and thus $X(t) = h\big(Y(t), Z^0(t), t\big)$ in a neighborhood of t_0. □

Remark 5.3 1. For the case $m = 0$ the above model is treated in [13].

2. In the general case the question of extending the solution to the whole time interval is reduced to that for the auxiliary differential equations (5.2) and (5.3), i.e. to growth conditions on the random vector fields. Russo and Vallois [8] considered the case $n = 1$ and

$$dX(t) \;=\; \sigma(X(t))\, dZ^0(t) + b(t, X(t))\, dV(t)$$

for a process V of bounded variation in full detail. They extended the Doss approach taking for h in (5.2) the flow given by

$$\begin{aligned} \frac{\partial h}{\partial z}(y, z) \;&=\; \sigma(h(y, z))\,, \\ h(y, 0) \;&=\; y\,. \end{aligned}$$

3. For $n = 1$ and random numbers $\sigma_0, \sigma_1, \dots, \sigma_m, \beta$ our method implies that the unique solution of the linear equation

$$\begin{aligned} dX(t) \;&=\; \sum_{i=0}^{m} \sigma_i X(t)\, dZ^i(t) + \beta X(t)\, dt\,, \\ X(0) \;&=\; X_0 \end{aligned}$$

is given by

$$X(t) = X_0 \exp\left\{\sum_{i=0}^{m} \sigma_i Z^i(t) - \frac{1}{2}\,\sigma_0^2\big[Z^0\big](t) + \beta t\right\}$$

and similarly for time dependent coefficients. Note that in the latter case the C^1-property of the σ_i is not really needed. It may be relaxed, e.g., to the condition that the summed order of differentiability of σ_i and Z_i is not less than 1.

4. The case of Gaussian driving processes $Z^0, \ldots, Z^m$ is considered in Coutin and Decreusefond [2] and other papers with methods of Malliavin calculus. The notion of Skorohod-type stochastic integral used there is different from the above version.

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